On (p,q)-rough paths

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Abstract: We extend the work of T. Lyons [Lyo98] and T. Lyons and Z. Qian [LQ02] to define integrals and solutions of differential equations along product of p and q rough paths, with $1/p+1/q > 1$. We use this to write an Itô formula at the level of rough paths, and to see that any rough path can always be interpreted as a product of a p-geometric rough path and a $p/2$-geometric rough path.

Keywords: rough path, controlled differential equation, Itô and Stratonovich integrals for semi-martingales, Hölder continuous paths

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1 Introduction

When x is a path of finite p-variation (for example, a $1/p$-Hölder continuous path) with values in a Banach space V (of finite or infinite dimension), integrals of type $\int f(x_s)dx_s$ or solutions of controlled differential equations of type $dy_t = A(y_t)dx_t$ can be constructed using the theory of rough paths [Lyo98, LQ02, Lej03a]. In order to do so, one needs to know a path $\mathbf{x}$, called a p-rough path, of finite p-variation with values in the space $T^{\lfloor p \rfloor}(V) = 1 \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes \lfloor p \rfloor}$, and such that the projection on V of $\mathbf{x}$ is x (we then say that $\mathbf{x}$ *lies above* x). This path $\mathbf{x}$ encodes the equivalent of the iterated integrals of x, and extends the notion of Chen series for irregular paths [Che58]. To consider $\int f(x)d\mathbf{x}$, the hypotheses on the smoothness of f is connected to the roughness of $\mathbf{x}$: if $\mathbf{x}$ is of finite p-variation, then f must be at least $\lfloor p \rfloor$ -times differentiable, with the derivative of order $\lfloor p \rfloor$ which is $(p - \lfloor p \rfloor + \varepsilon)$-Hölder continuous, $\varepsilon > 0$.

In this article, we deal with (p, q)-rough paths, that is a pair $(\mathbf{x}, \mathbf{h})$, where $\mathbf{x}$ is a p-rough path and $\mathbf{h}$ a q -rough path lying respectively above some paths x and h. This is generally not sufficient to properly define $\int f(x, h)d(\mathbf{x}, \mathbf{h})$, unless the cross-iterated integrals of x against h (and h against x) are specified. However, if $1/p + 1/q > 1$, with say $p > q$, these cross-iterated integrals can be canonically constructed as Young integrals, and there is a "natural" p-rough path $\mathbf{z}$ lying above $(\mathbf{x}, \mathbf{h})$. Thus, the integral $\int f(x, h)d(\mathbf{x}, \mathbf{h})$ can be defined as $\int f(z)d\mathbf{z}$ without ambiguity. Using this approach, one needs f to be at least $\lfloor p \rfloor$-times differentiable, with the derivative of order $\lfloor p \rfloor$ being $(p - \lfloor p \rfloor + \varepsilon)$-Hölder continuous, $\varepsilon > 0$.

In the case of (p, q)-rough paths with $p \in [2, 3)$ and $1/q + 1/p > 1$, we prove the existence of integrals of type $\int f(x, h)d(\mathbf{x}, h)$ under weaker regularity conditions on $h \mapsto f(x, h)$ than on $x \mapsto f(x, h)$, which appears to be natural when one considers the way integrals are constructed in the theory of rough paths. The continuity and local Lipschitzness of $(\mathbf{x}, h) \mapsto \int f(x, h)d(\mathbf{x}, h)$ is also proved. This serves as a basis for considering the solution of a system of differential equation of type

$$\begin{cases} dy_t = A(y_t, k_t)d\mathbf{x}_t + C(y_t, k_t)dh_t, \\ dk_t = B(y_t, k_t)dh_t. \end{cases} \tag{1}$$

Stochastic differential equations with a drift offer the canonical example of such a system. We will prove that the map $(\mathbf{x}, h) \to (\mathbf{y}, k)$ is continuous from the space of (p, q)-rough paths into itself, using the p-variation topology or some modulus type topology (e.g. the $1/p$-Hölder topology). Besides, the uniqueness and the continuity of this map, called the Itô map, follows from the same argument as the local Lipschitzness of the integral. This new understanding allows us to present a conceptually simplified proof of the continuity of the Itô map. With our construction of integrals or solutions of differential equations, (p, q)-rough paths are transformed into (p, q)-rough paths by integrating one-forms or solving differential equations. The notion of (p, q)-rough path appears as a natural extension of the notion of p-rough paths.

A martingale lifted to a p-rough path together with a bounded variation process is a natural example of a (p, q)-rough path ($q = 1$ here), that corresponds to a semi-martingale. When the bounded variation process is just the time, we see that we can consider differential equations with a drift term. Another example of a (p, q)-rough path is two fractional Brownian motions, with different Hurst indices.

One of our motivations to introduce the notion of (p, q)-rough paths is to see non-geometric rough path as a $(p, p/2)$-geometric rough path. When x is a smooth path in V, a rough path $\mathbf{x}$ with value in $T^p(V)$ can be easily constructed by defining the projection of $\mathbf{x}$ on $V^{\otimes k}$ as the k-th iterated integral $\int dx \otimes \cdots \otimes dx$. A geometric p-rough path is, by definition, an element of the closure of the space of such path in $T^p(V)$ with respect to a p-variation metric on $T^p(V)$. If we define a weak geometric p-rough path to be a $G^p(V)$-valued path of finite p-variation, we see that a geometric p-rough path is a weak geometric p-rough path, while a weak geometric p-rough path is geometric q-rough path for all $q > p$ [FV04]. If $\mathbf{x}$ is a weak geometric p-rough path and ψ is a path of finite $p/2$-variation with values in the symmetric part of $V^{\otimes 2}$, then $\mathbf{x} + \psi$ is a p-rough path. We will establish that this map $(\mathbf{x}, \psi) \to \mathbf{x} + \psi$ is actually a bijection from the space of p-rough paths onto the space of weak geometric $(p, p/2)$-rough paths, i.e. the space of pairs of p and $p/2$ weak geometric rough paths.

We express the integral along the non-geometric rough path $\mathbf{x} + \psi$ in term of an integral along a geometric rough path. This provides an alternative view of the Itô formula, even for processes that are not semi-martingales, which in this context shall be understood not as coming from the fact that we integrate irregular paths, but from the choice of a rough path which is not a geometric one. As a straightforward consequence, any integral of type $\int_0^\cdot f(x) d\mathbf{x}$, for $\mathbf{x}$ geometric or not, can be approximated by standard integrals in which the first level is

$$\int_0^\cdot f(x_s^n) dx_s^n + \int_0^\cdot \nabla f(x_s^n) d\psi_s^n,$$

where x^n is a smooth approximation of $\mathbf{x}$ on V, and ψ^n is a smooth approximation of ψ. Of course, a similar result holds for differential equations. Besides, this also means that the distinction between geometric rough paths and non-geometric rough paths (for $p<3$) is not as important as it seems.

Usually, an integral of type $\int f(x)d\mathbf{x}$ against a geometric rough path is seen as a Stratonovich integral by analogy to the Wong-Zakai theorem, because $\mathbf{x}\mapsto\int f(x)d\mathbf{x}$ is continuous and the fact that $\mathbf{x}$ may be approximated by piecewise smooth paths. Conversely, for a semi-martingale x, an Itô integral may be constructed using the theory of rough paths using p-rough paths that are not geometric p-rough paths. In this case, the $p/2$-rough path ψ in the decomposition $\mathbf{x}+\psi$ is given by $-\frac{1}{2}\langle x,x\rangle$ ($\langle x,x\rangle$ being the quadratic variation process of the semi-martingale x), and results of type Wong-Zakai can be given by exploiting the particular structure of this bracket [CL05].

2 Preliminaries

2.1 A bit of Algebra

When $(V,|\cdot|)$ is a separable Banach space, we equip $V\otimes V$ with a norm also denoted $|\cdot|$, such that for all $v,v'\in V$, $|v\otimes v'|\leq|v|\cdot|v'|$ and such that $\left|\sum_i v_i\otimes v_i'\right|=\left|\sum_i v_i'\otimes v_i\right|$. When V_1 and V_2 are two Banach spaces, $V_1\oplus V_2$ is again a Banach space with the norm $|(v_1,v_2)|=\max(|v_1|,|v_2|)$. We again equip $V_i\otimes V_j$, $i\neq j$, with a norm also denoted $|\cdot|$, such that for all $v_i,v_j\in V_i\oplus V_j$, $|v_i\otimes v_j|\leq|v_i|\cdot|v_j|$. We then put on $(V_1\oplus V_2)^{\otimes 2}$ the following norm: If $x=\sum_{i,j=1}^2 x_{i,j}$ with $x_{i,j}\in V_i\otimes V_j$, then $|x|=\max_{i,j}|x_{i,j}|$.

2.1.1 Some Lie Groups

We define on $T^2(V)=V\oplus V^{\otimes 2}$, a product

$$(v_1,v_2)\otimes(w_1,w_2)=(v_1+w_1,v_2+w_2+v_1\otimes w_1).$$

It makes $(T^2(V),\otimes)$ into a connected nilpotent Lie group, with $(0,0)$ as neutral element, and $(-v_1,v_1\otimes v_1-v_2)$ as inverse of (v_1,v_2). The Lie algebra of $(T^2(V),\otimes)$ is also $V\oplus V^{\otimes 2}$, with its Lie bracket given by

$$[(v_1,v_2),(w_1,w_2)]=(0,v_1\otimes w_1-w_1\otimes v_1).$$

The map

$$\begin{aligned}\exp:(V\oplus V^{\otimes 2},[\cdot,\cdot]) &\longrightarrow (T^2(V),\otimes)\\ (v_1,v_2) &\longrightarrow \left(v_1,v_2+\frac{1}{2}v_1^{\otimes 2}\right)\end{aligned}$$

defines a global isomorphism between the Lie algebra and the Lie group [Jac79a].

We denote by $\mathrm{Anti}(V^{\otimes 2})$ (resp. $\mathrm{Sym}(V^{\otimes 2})$) the set of elements of $V^{\otimes 2}$ which are antisymmetric (resp. symmetric). Then $(V \oplus \mathrm{Anti}(V^{\otimes 2}), [\cdot,\cdot])$ is a Lie subalgebra of $(V \oplus V^{\otimes 2}, [\cdot,\cdot])$, which makes

$$\begin{aligned} G^2(V) &= \exp\Big(V \oplus \mathrm{Anti}(V^{\otimes 2})\Big) \\ &= \{(v_1, v_2) \in T^2(V),\, v_2 - \frac{1}{2} v_1^{\otimes 2} \in \mathrm{Anti}(V^{\otimes 2})\} \end{aligned}$$

into a Lie subgroup of $T^2(V)$. The space $G^2(V)$ is actually the free nilpotent group of step 2 generated by V [Reu93].

On $T^2(V)$ (and hence $G^2(V)$), we define a dilation operator

$$\delta_\lambda(v_1, v_2) = (\lambda v_1, \lambda^2 v_2).$$

The space $T^2(V)$ obviously also enjoys a Banach space structure, with the norm $|(v_1, v_2)| = \max\{|v_1|, |v_2|\}$.

2.1.2 Homogeneous Norms on These Groups

A homogeneous norm [Ste70] on a Lie group G equipped with a dilation δ_λ is a map $\|\cdot\| : G \to \mathbb{R}^+$ which satisfies

(i) $\|g\| = 0$ if and only if $g = \exp(0)$,
(ii) for all $g \in G(\mathbb{R}^d)$ and $t \in \mathbb{R}$, $\|\delta_t g\| = |t| \, \|g\|$.

If moreover, for all $g, h \in G$, $\|g \otimes h\| \le \|g\| + \|h\|$, this homogeneous norm is said to be sub-additive. If for all $g \in G$, $\|g\| = \|g^{-1}\|$, this norm is said to be symmetric.

We define on $T^2(V)$ the following sub-additive homogeneous norm:

$$\begin{aligned} \|\cdot\| : T^2(V) &\longrightarrow \mathbb{R}^+ \\ (v_1, v_2) &\longrightarrow \max\left\{|v_1|, \sqrt{2|v_2|}\right\}. \end{aligned}$$

Note that when restricted to $G^2(V)$ it is actually symmetric, and that the symmetric sub-additive homogeneous norm $g \in T^2(V) \to \|g\| + \|g^{-1}\|$ is equivalent to $\|\cdot\|$ (See [FV04]).

The norm $\|\cdot\|$ induces a left-invariant metric d on $T^2(V)$ defined by $d(x, y) = \|x^{-1} \otimes y\|$.

2.2 Chen Series and p-Rough Paths

We denote by $C^{p\text{-var}}([0,T], E)$ the set of continuous function x from $[0,T]$ with values in a metric space (E, d) such that $\sup \sum_{i=0}^{n-1} d(x_{t_i}, x_{t_{i+1}})^p < \infty$, where the supremum is taken over all subdivision $(0 = t_0 \leq \cdots \leq t_n \leq T)$. This is the set of continuous E-valued functions of finite p-variation.

Let $x : [0,T] \to V$ be a continuous path of bounded variation. For such a path, we construct $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2) \in T^2(V)$ by

$$\mathbf{x}^1_{s,t} = x_t - x_s \text{ and } \mathbf{x}^2_{s,t} = \int_{s \leq r_1 \leq r_2 \leq t} dx_{r_1} \otimes dx_{r_2}. \tag{2}$$

It follows from the results of K.T. Chen (See for example [Che58, Reu93]) that $\mathbf{x}_{0,t} = \mathbf{x}_{0,s} \otimes \mathbf{x}_{s,t}$. In particular, $\mathbf{x}_{s,t} = \mathbf{x}_{0,s}^{-1} \otimes \mathbf{x}_{0,t}$, which shows that $(\mathbf{x}_{s,t})_{0 \leq s \leq t \leq 1}$ is easily obtained from the $(T^2(V), d)$-valued path $(\mathbf{x}_{0,t})_{t \in [0,T]}$. We then set $\mathbf{x}_0 = (x_0, 0)$ and $\mathbf{x}_t = \mathbf{x}_0 \otimes \mathbf{x}_{0,t}$. Nonetheless, we keep using the notation $\mathbf{x}_{s,t}$ for the element $\mathbf{x}_s^{-1} \otimes \mathbf{x}_t$. Another result of Chen is that for all $t \in [0,T]$, $\mathbf{x}_t$ belongs to $G^2(V) \subset T^2(V)$. We denote by $C^{1\text{-var}}([0,T], G^2(V))$ the set of paths $(\mathbf{x}_t)_{t \in [0,T]}$ lying above a continuous path of bounded variation and constructed by (2). It coincides with the set of paths of bounded variation with values in $G^2(V)$, by Theorem 1 in [Lyo98]; that justifies our notation.

A continuous path $\mathbf{x} : [0,T] \to T^2(V)$ is said to be of finite p-variation if

$$\|\mathbf{x}\|_{p\text{-var},T} = \sup \left(\sum_{i=0}^{n-1} \left\| \mathbf{x}_{t_i,t_{i+1}} \right\|^p \right)^{1/p} < \infty,$$

where the supremum is taken over all subdivision $(0 = t_0 \leq \cdots \leq t_n \leq T)$. We also define a p-variation distance, as in [Lyo98, LQ02]:

$$d_{p\text{-var}}(\mathbf{x}, \mathbf{y}) = \sup \left(\sum_{i=0}^{n-1} \left\| \mathbf{y}_{t_i,t_{i+1}} - \mathbf{x}_{t_i,t_{i+1}} \right\|^p \right)^{1/p}.$$

Note that from [FV04], $d_{p\text{-var}}$ induces the same topology than

$$\widetilde{d}_{p\text{-var}}(\mathbf{x}, \mathbf{y}) = \sup \left(\sum_{i=0}^{n-1} \|\mathbf{x}_{t_i,t_{i+1}}^{-1} \otimes \mathbf{y}_{t_i,t_{i+1}}\|^p \right)^{1/p}.$$

In the latter distance, the group structure of $T^2(V)$ is used, while in the first one, we use the vector space structure of $T^2(V)$. We can now define the notion of p-rough path, using slightly different words than in [LQ02].

Definition 1. Let $p \in [2,3)$. The set of geometric p-rough paths, denoted by $C^{0,p\text{-var}}([0,T], G^2(V))$, is the $d_{p\text{-var}}$-closure of $C^{1\text{-var}}([0,T], G^2(V))$. The set of weak geometric p-rough paths is denoted by $C^{p\text{-var}}([0,T], G^2(V))$ while the set of p-rough paths is denoted by $C^{p\text{-var}}([0,T], T^2(V))$.

For $p < q$, we observe that

$$C^{0,p\text{-var}}([0,T], G^2(V)) \subset C^{p\text{-var}}([0,T], G^2(V)) \subset C^{0,q\text{-var}}([0,T], G^2(V)),$$

and these inclusions are strict [FV04]. In particular, a p-rough path which takes its values in $G^2(V)$ is not necessarily a geometric p-rough path, but is a weak geometric p-rough path.

We say that ω is a *control* if
(i) $\omega : \{(s,t), 0 \leq s \leq t \leq 1\} \to \mathbb{R}^+$ is continuous.
(ii) ω is super-additive, i.e. $\forall\ s < t < u$, $\omega(s,t) + \omega(t,u) \leq \omega(t,u)$.
(iii) $\omega(t,t) = 0$ for all $t \in [0,1]$.

A path x is of finite p-variation if and only if there exists a control ω such that $\|x_{s,t}\|^p \leq \omega(s,t)$ for all $s < t$. For a given control ω, we define following [F05],

$$\begin{aligned}
\|\mathbf{x}\|_{p,\omega,T} &= \sup_{0 \leq s < t \leq T} \frac{\|\mathbf{x}_{s,t}\|}{\omega(s,t)^{1/p}}, \\
d_{p,\omega,T}(\mathbf{x}, \mathbf{y}) &= \sup_{0 \leq s < t \leq T} \max \left\{ \frac{\left|\mathbf{y}^1_{s,t} - \mathbf{x}^1_{s,t}\right|}{\omega(s,t)^{1/p}}, \frac{\left|\mathbf{y}^2_{s,t} - \mathbf{x}^2_{s,t}\right|}{\omega(s,t)^{2/p}} \right\}.
\end{aligned}$$

We will exclusively work with the distance $d_{p,\omega,T}$, being more general that the p-variation distance. We introduce the following sets:

$$\begin{aligned}
C^{p,\omega}([0,T], G^2(V)) &= \left\{ \mathbf{x} : [0,T] \to G^2(V),\ \|\mathbf{x}\|_{p,\omega,T} < \infty \right\}, \\
C^{p,\omega}([0,T], T^2(V)) &= \left\{ \mathbf{x} : [0,T] \to T^2(V),\ \|\mathbf{x}\|_{p,\omega,T} < \infty \right\}.
\end{aligned}$$

$C^{p,\omega}([0,T], T^2(V))$ is the set of p-rough path whose p-variation controlled by ω.

Finally, we say that a rough path $\mathbf{x}$ *lies above* a path x if $\mathbf{x}^1_t = x_t$. By convention, rough paths are denoted by bold letters and the corresponding italic letters are used to denote the paths they lie above.

3 Integrating Signals made of p-Rough Paths and q-rough Paths

3.1 Lipschitz Functions, Differential Forms

When V, W are two vector spaces, we define by $L(V,W)$ the set of linear maps from V into W. Let $(X, |\cdot|)$ be a Banach vector space. For $\gamma > 0$, we set $[\gamma] = \gamma - 1$ if $\gamma \in \mathbb{N}$ and $[\gamma] = \lfloor \gamma \rfloor$, the integer part of γ, if $\gamma \notin \mathbb{N}$.

Definition 2. For $\gamma > 0$, f belongs to $\mathrm{Lip}(\gamma, V \to X)$ if and only if

1. $f : V \to X$ is $[\gamma]$-times differentiable. By definition, this means that there exists $d^j f : V \to L(V^{\otimes j}, X)$, $j = 0, \cdots, [\gamma]$ (with $d^0 f = f$ by convention) such that for all $j = 0, \cdots, [\gamma] - 1$, $x, y \in V$,

$$d^j f(y)(v_j) - d^j f(x)(v_j) = \int_0^1 d^{j+1} f(x + t(y-x))((y-x) \otimes v_j) dt.$$

2. $d^j f$ is bounded by K, for all $j = 0, \cdots, [\gamma]$.

3. $d^{[\gamma]} f$ is $(\gamma - [\gamma])$-Hölder, with Hölder constant K, i.e. for all $x \neq y \in V$,

$$\frac{|d^{[\gamma]} f(x) - d^{[\gamma]} f(y)|}{|x-y|^{\gamma - [\gamma]}} \leq K.$$

The smallest constant K for which these equations are satisfied is called the *Lipschitz norm of* f and is denoted by $\|f\|_{\mathrm{Lip}(\gamma, V \to X)}$ or $\|f\|_{\mathrm{Lip}}$ when the context is clear.

In particular, when $X = L(V, W)$, we have a notion of γ-Lipschitz one-form from V to $L(V, W)$.

Example 1. If $g : \mathbb{R}^N \to \mathbb{R}^M$ belongs to $\mathrm{Lip}(1 + \gamma, \mathbb{R}^N \to \mathbb{R}^M)$, then dg belongs to $\mathrm{Lip}(\gamma, \mathbb{R}^N \to L(\mathbb{R}^N, \mathbb{R}^M))$.

We extend now this definition to mixed (γ, η)-Lipschitz norms, where γ and η are positive reals.

Definition 3. For $\gamma, \eta > 0$, f belongs to $\mathrm{Lip}((\gamma, \eta), V_1 \times V_2 \to X)$ if and only if

1. There exists $d^{i,j} f : V \to L(V_1^{\otimes i} \otimes V_2^{\otimes j}, X)$, $0 \leq i \leq [\gamma]$, $0 \leq j \leq [\eta]$ (with $d^{0,0} f = f$ by convention) such that for all $0 \leq i \leq [\gamma] - 1$, $0 \leq j \leq [\eta] - 1$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, $(v_{1,i}, v_{2,j}) \in V_1^{\otimes i} \times V_2^{\otimes j}$,

$$\begin{aligned} d^{i,j} f(x_1, y_1)&(v_{1,i}, v_{2,j}) - d^{i,j} f(x_1, y_2)(v_{1,i}, v_{2,j}) \\ &= \int_0^1 d^{i,j+1} f(x_1, y_1 + t(y_2 - y_1))(v_{1,i}, (y_2 - y_1) \otimes v_{2,j}) dt, \end{aligned}$$

$$\begin{aligned} d^{i,j} f(x_1, y_1)&(v_{1,i}, v_{2,j}) - d^{i,j} f(x_2, y_1)(v_{1,i}, v_{2,j}) \\ &= \int_0^1 d^{i+1,j} f(x_1 + t(x_2 - x_1), y_1)((x_2 - x_1) \otimes v_{1,i}, v_{2,j}) dt. \end{aligned}$$

2. $d^{i,j}f$ are bounded by K, $0 \le i \le [\gamma]$, $0 \le j \le [\eta]$.
3. $d^{[\gamma],j}f$, $0 \le j \le [\eta]$ is $(\gamma - [\gamma])$-Hölder, with Hölder constant K.
4. $d^{i,[\eta]}f$, $0 \le i \le [\gamma]$ is $(\eta - [\eta])$-Hölder, with Hölder constant K.

The smallest constant K for which these equations are satisfied is called the *Lipschitz norm of* f and is denoted by $\|f\|_{\mathrm{Lip}((\gamma,\eta),V_1\times V_2\to X)}$ or $\|f\|_{\mathrm{Lip}}$ when the context is clear.

We will be particularly interested to the cases $X = L(V_1, W)$, $X = L(V_2, W)$ or $X = L(V_1 \oplus V_2, W)$, and we restrict ourselves to the case $p \in [2, 3)$.

To integrate a one-form $f \in \mathrm{Lip}(\gamma - 1, V \to L(V, W))$, $\gamma > p$, along a p-rough path $\mathbf{x}$, the differential form is "lifted" into

$$\mathsf{f} : V \longrightarrow L(T^2(V), T^2(W))$$

such that $\mathsf{f}(x)(v_1, v_2) = (f(x)(v_1) + d^1 f(x)(v_2)) + (f(x) \otimes f(x))(v_2)$.

Given $\mathbf{x} \in C^{p\text{-var}}([0,T], T^2(V))$ with $p < \gamma$, it is now possible to construct $\int_0^t f(x_s)d\mathbf{x}_s$ as the limit

$$\mathfrak{M}(\mathbf{z})_{0,t} = \lim_{\delta\to 0} \mathbf{z}_{t_0^\delta,t_1^\delta} \otimes \mathbf{z}_{t_1^\delta,t_2^\delta} \otimes \cdots \otimes \mathbf{z}_{t^\delta_{k^\delta-2},t^\delta_{k^\delta-1}} \otimes \mathbf{z}_{t^\delta_{k^\delta-1},t^\delta_{k^\delta}}, \tag{3}$$

where $\{ t_0^\delta \le \ldots \le t^\delta_{k^\delta} \}$ is a partition of $[0, t]$ whose mesh decreases to 0 as $\delta \to 0$ and

$$\mathbf{z}_{s,t} = \mathsf{f}(x_s)\mathbf{x}_{s,t}.$$

The element $\mathbf{z}$ is not a rough path, but it is called an *almost rough path*, since

$$\text{for any } 0 \le s \le u \le t,\ |\mathbf{z}_{s,t} - \mathbf{z}_{s,u} \otimes \mathbf{z}_{u,t}| \le C\omega(s,t)^\theta$$

for some constant $\theta > 1$ and some constant C, if $\mathbf{x}$ is of finite p-variation controlled by ω. Here, $|\cdot|$ is the vector space norm on $T^2(V)$ given by $|v_1 + v_2| = \max\{|v_1|, |v_2|\}$ or any other equivalent norm. The existence of the limit in (3) follows from an adaptation due to T. Lyons [Lyo98] of the arguments used by L.C. Young in [You36] to construct its integral.

We now want to generalise this argument to integrate one-forms with respect to product of a p-rough path and a q-rough path, for $1/p + 1/q > 1$.

3.2 Integration along a One-form, Introduction

Still in the case $p \in [2, 3)$, we now consider $q \geq 1$ such that $1/p + 1/q > 1$ (which implies $q < 2$). Let $(\mathbf{x}, h)$ be an element of $C^{p\text{-var}}([0, T], T^2(V_1)) \times C^{q\text{-var}}([0, T], V_2)$, for two Banach spaces V_1, V_2. We will call such an element a (p, q)-rough path. We simplify the notation and denote by $C^{(p,q)\text{-var}}([0, T], T^2(V_1) \times V_2)$ this set, and also

$$C^{(p,q),\omega}([0, T], T^2(V_1) \times V_2) = C^{p,\omega}([0, T], T^2(V_1)) \times C^{q,\omega}([0, T], V_2)$$

which we equip with the norm $\|(\mathbf{x}, h)\|_{(p,q),\omega,T} = \max\{\|\mathbf{x}\|_{p,\omega,T}, \|h\|_{q,\omega,T}\}$. As $2/q > 1$ and $1/p + 1/q > 1$, $\int dh \otimes dh$, $\int d\mathbf{x}^1 \otimes dh$ and $\int dh \otimes d\mathbf{x}^1$ are well defined Young integrals.

Lemma 1 ([LQ97, LQ02]). *For any control ω, the map*

$$\begin{aligned} \Pi : C^{(p,q),\omega}([0, T], T^2(V_1) \times V_2) &\longrightarrow C^{p,\omega}([0, T], T^2(V_1 \oplus V_2)) \\ (\mathbf{x}, h) &\longrightarrow \Pi(\mathbf{x}, h) \end{aligned}$$

where $\Pi(\mathbf{x}, h)^1_t = \mathbf{x}^1_t + h_t$ and $\Pi(\mathbf{x}, h)^2_t = \mathbf{x}^2_t + \int_{0<u<v<t} d(h_u \otimes dh_u + d\mathbf{x}^1_u \otimes dh_v + dh_u \otimes d\mathbf{x}^1_v)$ is well defined and continuous.

One can then integrate one forms $V_1 \oplus V_2 \to L(V_1 \oplus V_2, W)$ which are smooth enough along $(\mathbf{x}, h)$ (by integrating along $\Pi(\mathbf{x}, h)$). Nonetheless, as we will see, doing it this way, we are imposing stronger smoothness conditions on the one-form than needed.

One of the most important result of [Lyo98] is to give a procedure, which we denote here as a map $\mathfrak{M}$ defined in (3), to construct a rough path from an almost multiplicative functional.

Lemma 2. *If $\widetilde{\mathbf{x}}$ (resp. $\widetilde{h}$) is an almost multiplicative functional of finite p-variation (resp. q-variation), such that $\mathfrak{M}(\widetilde{\mathbf{x}}) = \mathbf{x}$ and $\mathfrak{M}(\widetilde{h}) = h$, then $\mathfrak{M}(\widetilde{\mathbf{x}}^1 + \widetilde{h}, \widetilde{\mathbf{x}}^2) = \Pi(\mathbf{x}, h)$.*

Proof. By Young's estimates, $\mathfrak{M}(\mathbf{x}^1 + h, \mathbf{x}^2) = \Pi(\mathbf{x}, h)$; Moreover, $|\mathbf{x}^1_{s,t} + h_{s,t} - \widetilde{\mathbf{x}}^1_{s,t} - \widetilde{h}_{s,t}| + |\mathbf{x}^2_{s,t} - \widetilde{\mathbf{x}}^2_{s,t}|$ is by assumption bounded by $C\omega(s, t)^\theta$, where ω controls the p-variation of $\widetilde{\mathbf{x}}$ and and q-variation of $\widetilde{\mathbf{y}}$, and $\theta > 1$. Therefore, by Theorem 3.3.1 in [Lyo98], we obtain our result. ∎

3.3 Construction and Continuity

In this Section, we still consider that $p \in [2, 3)$, $q \geq 1$ and $1/p + 1/q > 1$.

We are now going to construct a notion of integral $(\mathbf{x}, h) \mapsto \int f(x, h) d\mathbf{x}$, where f is a map from $V_1 \times V_2$ into linear functionals from V_1 into W and x is the path lying below $\mathbf{x}$, that is the projection of $\mathbf{x}$ on V_1.

Proposition 1. *Let* $(\mathbf{x},h)$ *be in* $C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2)$ *and* f *be a* $\mathrm{Lip}((\gamma-1,\kappa-1),V_1\times V_2\to L(V_1,W))$ *one-form with*

$$\gamma>p \text{ and } \kappa>\frac{qp+p-q}{p}. \tag{4}$$

Then $\mathbf{z}=(\mathbf{z}_{s,t})_{0\leq s\leq t\leq T}$ *given by*

$$\begin{aligned}\mathbf{z}^1_{s,t}&=f(\mathbf{x}^1_s,h_s)\mathbf{x}^1_{s,t}+d^{1,0}f(\mathbf{x}^1_s,h_s)\mathbf{x}^2_{s,t},\\ \mathbf{z}^2_{s,t}&=f(\mathbf{x}^1_s,h_s)\otimes f(\mathbf{x}^1_s,h_s)\mathbf{x}^2_{s,t}\end{aligned} \tag{5}$$

is an almost multiplicative functional. Let $\int f(x,h)d\mathbf{x}=\mathfrak{M}(\mathbf{z})$ *be the corresponding rough path. Then the map* $(\mathbf{x},h)\mapsto\int f(\mathbf{x},h)d\mathbf{x}$ *is continuous from* $C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2)$ *into* $C^{p,\omega}([0,T],T^2(W))$*. Moreover,*

$$\left\|\int f(x,h)d\mathbf{x}\right\|^p_{p,\omega,T}\leq K\max\left\{\|\mathbf{x}\|^p_{p,\omega,T},\|h\|^q_{q,\omega,T}\right\}, \tag{6}$$

where K *depends only on* $\|f\|_{\mathrm{Lip}}$*,* γ*,* κ*,* p*,* q*,* C_ω*, where* C_ω *is any constant bounding above* $\max\left\{\|\mathbf{x}\|^p_{p,\omega,T},\|h\|^q_{q,\omega,T}\right\}\omega(0,T)$*.*

Proof. We first need to check that for all $0\leq s\leq u\leq t\leq T$, one has $\left|\mathbf{z}^1_{s,u}-\mathbf{z}^1_{s,t}-\mathbf{z}^1_{t,u}\right|\leq C\omega(s,t)^\theta$ for some $C>0$, $\theta>1$. We define $\widetilde{\omega}=\max\{\|\mathbf{x}\|^p_{p,\omega,T},\|h\|^q_{q,\omega,T}\}\omega$, so that for all $s<t$, $\|\mathbf{x}_{s,t}\|^p$ and $|h_{s,t}|^q$ are bounded by $\widetilde{\omega}(s,t)$. For all $0\leq s\leq u\leq t\leq T$,

$$\begin{aligned}\mathbf{z}^1_{s,u}-\mathbf{z}^1_{s,t}-\mathbf{z}^1_{t,u}&=(f(\mathbf{x}^1_s,h_s)-f(\mathbf{x}^1_t,h_t))\mathbf{x}^1_{t,u}+d^{1,0}f(\mathbf{x}^1_s,h_s)\mathbf{x}^1_{s,t}\otimes\mathbf{x}^1_{t,u}\\&\quad+(d^{1,0}f(\mathbf{x}^1_s,h_s)-d^{1,0}f(\mathbf{x}^1_t,h_t))\mathbf{x}^2_{t,u}\\&=\left\{\int_0^1(d^{1,0}f(\mathbf{x}^1_s,h_s)-d^{1,0}f(\mathbf{x}^1_s+a\mathbf{x}^1_{s,t},h_s))da\right\}(\mathbf{x}^1_{s,t}\otimes\mathbf{x}^1_{t,u})\\&\quad+(f(\mathbf{x}^1_t,h_s)-f(\mathbf{x}^1_t,h_t))\mathbf{x}^1_{t,u}\\&\quad+(d^{1,0}f(\mathbf{x}^1_s,h_s)-d^{1,0}f(\mathbf{x}^1_t,h_t))\mathbf{x}^2_{t,u}.\end{aligned} \tag{7}$$

We therefore see that, if $f\neq0$,

$$\begin{aligned}\frac{\left|\mathbf{z}^1_{s,u}-\mathbf{z}^1_{s,t}-\mathbf{z}^1_{t,u}\right|}{\|f\|_{\mathrm{Lip}}}&\leq 2\widetilde{\omega}(s,u)^{\frac{\gamma}{p}}+\widetilde{\omega}(s,u)^{\frac{\kappa-1}{q}+\frac{1}{p}}+\widetilde{\omega}(s,u)^{\frac{\kappa-1}{q}+\frac{2}{p}}\\&\leq C(\gamma,\kappa,p,q,C_\omega)\widetilde{\omega}(s,u)^\theta,\end{aligned}$$

where $\theta=\min\left\{\frac{\gamma}{p},\frac{\kappa-1}{q}+\frac{1}{p}\right\}>1$, and C_ω is any constant bounding above

$\widetilde{\omega}(0,T)$. Let us deal with the second level:

$$\begin{aligned}\mathbf{z}^2_{s,u}-(\mathbf{z}_{s,t}\otimes\mathbf{z}_{t,u})^2 &= (f(\mathbf{x}^1_s,h_s)^{\otimes 2}-f(\mathbf{x}^1_t,h_t)^{\otimes 2})\mathbf{x}^2_{t,u}\\ &\quad +f(\mathbf{x}^1_s,h_s)\otimes(f(\mathbf{x}^1_s,h_s)-f(\mathbf{x}^1_t,h_t))\mathbf{x}^1_{s,t}\otimes\mathbf{x}^1_{t,u}\\ &\quad +(f(\mathbf{x}^1_s,h_s)\otimes d^{1,0}f(\mathbf{x}^1_t,h_t))\mathbf{x}^1_{s,t}\otimes\mathbf{x}^2_{t,u}\\ &\quad +(d^{1,0}f(\mathbf{x}^1_s,h_s)\otimes f(\mathbf{x}^1_s,h_s))\mathbf{x}^2_{s,t}\otimes\mathbf{x}^1_{t,u}\\ &\quad +(d^{1,0}f(\mathbf{x}^1_s,h_s)\otimes d^{1,0}f(\mathbf{x}^1_t,h_t))\mathbf{x}^2_{s,t}\otimes\mathbf{x}^2_{t,u}.\end{aligned}$$

Bounding line by line, we easily obtain that

$$|\mathbf{z}^1_{s,u}-\mathbf{z}^1_{s,t}-\mathbf{z}^1_{t,u}|+|\mathbf{z}^2_{s,u}-(\mathbf{z}_{s,t}\otimes\mathbf{z}_{t,u})^2|\leq C(\max\{\|\mathbf{x}\|^p_{p,\omega,T},\|h\|^q_{q,\omega,T}\}\omega(s,u))^\theta,$$

where C depends on $\|f\|_{\mathrm{Lip}},\gamma,\kappa,p,q$ and C_ω. Moreover,

$$\begin{aligned}\|\mathbf{z}_{s,t}\| &\leq \|f\|_{\mathrm{Lip}}\|\mathbf{x}\|_{p,\omega,T}\,\omega(s,t)^{1/p}\\ &\leq \|f\|_{\mathrm{Lip}}\max\{\|\mathbf{x}\|^p_{p,\omega,T},\|h\|^q_{q,\omega,T}\}\omega(s,t)^{1/p}.\end{aligned}$$

Therefore, by Corollary 3.1.1 in [Lyo98], we obtain the bound (6).

The continuity follows as a corollary of Theorem 3.2.2 in [LQ02]. ∎

The following proposition has a similar and easier proof, that we omit.

Proposition 2. *Let* $(\mathbf{x},h)$ *in* $C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2)$ *and* g *be a* $\mathrm{Lip}((\alpha-1,\beta-1),V_1\times V_2\to L(V_2,W))$ *one-form with*

$$\alpha>\frac{pq+q-p}{q}\text{ and }\beta>q.$$

Then $z_{s,t}=g(\mathbf{x}_s,h_s)h_{s,t}$ *is an almost multiplicative functional. Let* $\int g(\mathbf{x},h)dh=\mathfrak{M}(z)$ *be the corresponding rough path. Then* $(\mathbf{x},h)\mapsto\int g(\mathbf{x},h)dh$ *is continuous from* $C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2)$ *into* $C^{q,\omega}([0,T],W)$*. Moreover,*

$$\left\|\int g(x,h)dh\right\|^q_{q,\omega,T}\leq K\max\left\{\|\mathbf{x}\|^p_{p,\omega,T},\|h\|^q_{q,\omega,T}\right\}$$

where K *depends only on* $\|g\|_{\mathrm{Lip}}$, α, β, p, q, C_ω *where* C_ω *is any constant bounding above* $\max\left\{\|\mathbf{x}\|^p_{p,\omega,T},\|h\|^q_{q,\omega,T}\right\}\omega(0,T)$.

In particular, we can integrate one forms from $V_1\times V_2$ to $L(V_1\oplus V_2,W_1\oplus W_2)$ along a (p,q)-rough path, to obtain a new (p,q)-rough path. The next proposition shows that the integrals that we have introduced coincide with the one in [Lyo98, LQ02], when one lifts these (p,q)-rough paths to p-rough paths with the map Π.

Proposition 3. *Let φ be a* $\mathrm{Lip}(\gamma-1, V_1\times V_2\to L(V_1\oplus V_2, W))$ *one-form with $\gamma>p$, and $(\mathbf{x},h)$ in $C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2)$. The map φ induces a* $\mathrm{Lip}((\gamma-1,1),V_1\times V_2\to L(V_1,W))$ *one-form*

$$\begin{array}{rcl}\varphi_1 : V_1\times V_2 & \longrightarrow & L(V_1,W)\\ (v_1,v_2) & \to & (dv_1\longrightarrow \varphi(v_1,v_2)(dv_1,0))\end{array}$$

and a $\mathrm{Lip}((\gamma-1,1),V_1\times V_2\to L(V_2,W))$ *one-form*

$$\begin{array}{rcl}\varphi_2 : V_1\times V_2 & \longrightarrow & L(V_2,W)\\ (v_1,v_2) & \longrightarrow & (dv_2\to \varphi(v_1,v_2)(0,dv_2)).\end{array}$$

Then, $\Pi\left(\int\varphi_1(x,h)d\mathbf{x},\int\varphi_2(x,h)dh\right)=\int\varphi(\Pi(\mathbf{x},h))d\Pi(\mathbf{x},h)$.

Proof. By Lemma 2 and the previous two propositions, $\Pi\left(\int\varphi_1(x,h)d\mathbf{x},\int\varphi_2(x,h)dh\right)$ is the unique rough path associated to the almost multiplicative functional

$$\begin{aligned}\mathbf{z}^1_{s,t}&=\varphi(\mathbf{x}^1_s,h_s)(\mathbf{x}^1_{s,t}+h_{s,t})+d^{1,0}\varphi(\mathbf{x}^1_s,h_s)\mathbf{x}^2_{s,t},\\ \mathbf{z}^2_{s,t}&=\varphi(\mathbf{x}^1_s,h_s)\otimes\varphi(\mathbf{x}^1_s,h_s)\mathbf{x}^2_{s,t}.\end{aligned}$$

On the other hand, if we let $\mathbf{y}=\Pi(\mathbf{x},h)$,

$$\begin{aligned}{\mathbf{z}^1_{s,t}}'&=\varphi(y_s)\mathbf{y}^1_{s,t}+d^{1,0}\varphi(y_s)\mathbf{y}^2_{s,t},\\ {\mathbf{z}^2_{s,t}}'&=\varphi(y_s)\otimes\varphi(y_s)\mathbf{y}^2_{s,t},\end{aligned}$$

is the almost multiplicative functional which constructs $\int\varphi(\Pi(\mathbf{x},h))d\Pi(\mathbf{x},h)$.
As

$$\begin{aligned}\mathbf{z}^1_{s,t}-{\mathbf{z}^1_{s,t}}' &= d^{1,0}\varphi(\mathbf{x}^1_s,h_s)\left(\int_{s<u<v<t}dh_u\otimes dh_v+d\mathbf{x}^1_u\otimes dh_v+dh_u\otimes d\mathbf{x}^1_v\right),\\ \mathbf{z}^2_{s,t}-{\mathbf{z}^2_{s,t}}' &= \varphi(\mathbf{x}^1_s,h_s)^{\otimes 2}\left(\int_{s<u<v<t}dh_u\otimes dh_v+d\mathbf{x}^1_u\otimes dh_v+dh_u\otimes d\mathbf{x}^1_v\right)\end{aligned}$$

the Young estimates implies that $\left|\mathbf{z}^1_{s,t}-{\mathbf{z}^1_{s,t}}'\right|+\left|\mathbf{z}^2_{s,t}-{\mathbf{z}^2_{s,t}}'\right|\le C(\omega(s,t)^{2/q}+\omega(s,t)^{1/q+1/p})$. The proof is then finished by Theorem 3.3.1 in [Lyo98].

3.4 A Remark on the Continuity

It is now well known that a family of rough paths $\mathbf{x}^\varepsilon$ in $C^{p,\omega}([0,T];V)$ whose p-norm $\|\mathbf{x}^\varepsilon\|_{p,\omega,T}$ is uniformly bounded in ε has a convergent subsequence in $C^{q,\omega}([0,T];V)$ provided that $(x^\varepsilon_0)_{\varepsilon>0}$ is bounded.

The generalisation of this result to (p,q)-rough paths is immediate.

Proposition 4. *Let $(\mathbf{x}^\epsilon, h^\epsilon)_{\epsilon>0}$ be a sequence of (p,q)-rough paths in the space $C^{(p,q),\omega}([0,T];T^2(V_1)\times V_2)$ such that the sequences $(x_0^\epsilon)_{\epsilon>0}$, $(h_0^\epsilon)_{\epsilon>0}$, $(\|\mathbf{x}^\epsilon\|_{p,\omega,T})_{\epsilon>0}$ and $(\|h^\epsilon\|_{q,\omega,T})_{\epsilon>0}$ are bounded. Then there exists a (p,q)-rough path $(\mathbf{x},h)$ which is limit of a a subsequence of $(\mathbf{x}^\epsilon, h^\epsilon)_{\epsilon>0}$ in $C^{(p',q'),\omega}([0,T];T^2(V_1)\times V_2)$ for any $p'>p$, $q'>q$.*

Yet it may happens that a sequence $(\mathbf{x}^\epsilon, h^\epsilon)_{\epsilon>0}$ does not satisfy the conditions of the Proposition 4, while $(\Pi(\mathbf{x}^\epsilon, h^\epsilon))_{\epsilon>0}$ is uniformly bounded in p-variation and then has a convergent subsequence in p'-variation for any $p'>p$. It has to be noted that in this case, any possible limit of $(\Pi(\mathbf{x}^\epsilon, h^\epsilon))_{\epsilon>0}$ lies above a limit of $(x^\epsilon, h^\epsilon)_{\epsilon>0}$. In this case, one has then to carefully study the limit behavior of $\Pi(\mathbf{x}^\epsilon, h^\epsilon)$ at the second level, and not only the one of $\mathbf{x}^\epsilon$, and thus the limit of the terms $\int dx^\epsilon \otimes h^\epsilon$, $\int dh^\epsilon \otimes dx^\epsilon$ and $\int dh^\epsilon \otimes dh^\epsilon$.

The homogenisation theory provides a non-trivial example of such a phenomena, as shown in [Lej02, LL03]. A semi-martingale $M+V$ represents the canonical example of a path lying below of (p,q)-rough paths $(\mathbf{M}, V)$, with $p>2$ and $q=1$. Let us consider some functions σ and b defined on $\mathbb{R}^d$ that are periodic and smooth enough, where $b(x)\in\mathbb{R}^d$ and σ takes its value in the space of symmetric matrix of size $d\times d$ and is uniformly elliptic. Let X be the solution of the stochastic differential equation $X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds$, where B is a d-dimensional Brownian motion. Indeed, it is classical that under some averaging condition on b [BLP78], $X^\epsilon = \epsilon X_{\cdot/\epsilon^2}$ converges to some process $\overline{X} = \overline{\sigma}\widetilde{B}_t$ in the space of continuous functions, where $\widetilde{B}$ is a Brownian motion, $\overline{\sigma}$ is a constant matrix. In addition, as shown in [LL03], the convergence holds also in p-variation for any $p>2$. Yet this convergence does not hold in the sense of (p,q)-rough paths, since the drift term of X^ϵ, which is $\epsilon\int_0^{\cdot/\epsilon^2} b(X_s)ds$, converges to a martingale and cannot be bounded in q-variation for $q\leq 2$. If $A_t(X^\epsilon)$ is the Lévy area of X^ϵ up to time t, it was proved in [Lej02, LL03] that for all $p>1$, $A_t(X^\epsilon)$ converges in p-variation to $A_t(\overline{X}) + ct$ where c is an antisymmetric matrix, as one can expect. This means that in this case, knowing only the limit $\overline{X}$ of $(X^\epsilon)_{\epsilon>0}$ is not sufficient to know the limits of $\int f(X^\epsilon)dX^\epsilon$ and of SDE driven by X^ϵ.

Finally, let us note that Proposition 4 has its counterpart in the semi-martingale context with the condition UCV (Uniformly Controlled Variation) (see [KP96a] for a review on the subject). Roughly speaking, a tight family semi-martingales with canonical decomposition $M^\epsilon + V^\epsilon$ whose local martingales M^ϵ have uniformly bounded brackets in probability and the 1-variation of V^ϵ are uniformly bounded in probability, converges along a subsequence to a semi-martingale with canonical decomposition $M+V$, where (M,V) is the limit of (M^ϵ, V^ϵ). In this case, stochastic integrals as well as SDE driven by $M^\epsilon + V^\epsilon$ also converge to stochastic integrals and SDE driven by the limit

$M+V$. Indeed, if $(M^\epsilon+V^\epsilon)_{\epsilon>0}$ satisfies the condition UCV, then $(M^\epsilon)_{\epsilon>0}$ is also bounded in p-variation for any $p>2$, and the conditions of Proposition 4 are also satisfied, as shown in [CL05].

This result may be generalised to processes generated by divergence form operators, that belongs to the class of Dirichlet processes, which is more general than semi-martingales. Loosely speaking, a Dirichlet process is the sum of a local martingale and a process with zero quadratic variation (yet several definitions exists for this type of process). For processes generated by divergence-form operators, we proved in [Lej03b, Lej03c] (see also [BHL02] on the application of the theory of rough paths to these processes) that the condition UTD (which is in some sense the equivalent of the condition UCV for semi-martingales) is sufficient to ensure the convergence of the Lévy areas to the Lévy area of the limit, and give then the possibility to interchange limits and integrals.

3.5 Lipschitzness of the Integral Map

By considering one-forms which are a bit smoother, we will show that the maps $(\mathbf{x},h)\mapsto\int f(x,h)d\mathbf{x}$ and $(\mathbf{x},h)\mapsto\int g(x,h)dh$ are actually locally Lipschitz, again in the case $p\in[2,3)$, $q\geq 1$ and $1/p+1/q>1$.

Proposition 5. *Let f be a $\mathrm{Lip}((\gamma,\kappa),V_1\times V_2\to L(V_1,W))$ one-form with*

$$\gamma>p,\ \text{ and } \kappa>\frac{qp+p-q}{p}.$$

Then the map $(\mathbf{x},h)\mapsto\int f(x,h)d\mathbf{x}$ is locally Lipschitz continuous from $(C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2),\|\cdot\|_{(p,q),\omega,T})$ into $(C^{p,\omega}([0,T],T^2(W)),\|\cdot\|_{p,\omega,T})$.

We start with a technical lemma.

Lemma 3. *Let $(\mathbf{x},h)$ and $(\widehat{\mathbf{x}},\widehat{h})$ be in $C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2)$,*

$$R>\max\left\{\|\mathbf{x}\|_{p,\omega,T},\|h\|_{q,\omega,T},\|\widehat{\mathbf{x}}\|_{p,\omega,T},\|\widehat{h}\|_{q,\omega,T}\right\}$$

and $\varepsilon=\max\left\{d_{p,\omega,T}(\mathbf{x},\widehat{\mathbf{x}}),d_{p,\omega,T}(h,\widehat{h})\right\}$.
We define $\mathbf{z}=(\mathbf{z}_{s,t})_{0\leq s\leq t\leq T}$ and $\widehat{\mathbf{z}}=(\widehat{\mathbf{z}}_{s,t})_{0\leq s\leq t\leq T}$ by

$$\left\{\begin{array}{r}\mathbf{z}^1_{s,t}=f(\mathbf{x}^1_s,h_s)\mathbf{x}^1_{s,t}+d^{1,0}f(\mathbf{x}^1_s,h_s)\mathbf{x}^2_{s,t},\\ \mathbf{z}^2_{s,t}=f(\mathbf{x}^1_s,h_s)\otimes f(\mathbf{x}^1_s,h_s)\mathbf{x}^2_{s,t},\end{array}\right.$$
$$\left\{\begin{array}{r}\widehat{\mathbf{z}}^1_{s,t}=f(\widehat{\mathbf{x}}^1_s,\widehat{h}_s)\widehat{\mathbf{x}}^1_{s,t}+d^{1,0}f(\widehat{\mathbf{x}}^1_s,\widehat{h}_s)\widehat{\mathbf{x}}^2_{s,t},\\ \widehat{\mathbf{z}}^2_{s,t}=f(\widehat{\mathbf{x}}^1_s,\widehat{h}_s)\otimes f(\widehat{\mathbf{x}}^1_s,\widehat{h}_s)\widehat{\mathbf{x}}^2_{s,t},\end{array}\right.$$

so that $\int f(x,h)d\mathbf{x}=\mathfrak{M}(\mathbf{z})$ *and* $\int f(\widehat{x},\widehat{h})d\widehat{\mathbf{x}}=\mathfrak{M}(\widehat{\mathbf{z}})$. *Then, for all* $0\leq s\leq t\leq u\leq T$, *we have*

$$\left|(\mathbf{z}^1_{s,u}-(\mathbf{z}^1_{s,t}+\mathbf{z}^1_{t,u}))-(\widehat{\mathbf{z}}^1_{s,u}-(\widehat{\mathbf{z}}^1_{s,t}+\widehat{\mathbf{z}}^1_{t,u}))\right|\leq K\varepsilon\omega(s,u)^\theta,$$

where $\theta>1$ *and* K *depends only on* $R,\omega(0,T),p,q,\gamma,\kappa$ *and* $\|f\|_{\mathrm{Lip}}$.

Proof. In this proof and the next one, K^i, $i\geq 0$ will denote some constants which may depend on $R,\omega(0,T),p,q,\gamma,\kappa$ and $\|f\|_{\mathrm{Lip}}$. We have already observed for all $s\leq u\leq t$,

$$\begin{aligned}\mathbf{z}^1_{s,u}-\mathbf{z}^1_{s,t}-\mathbf{z}^1_{t,u} &= \int_0^1((d^{1,0}f(\mathbf{x}^1_s,h_s)-d^{1,0}f(\mathbf{x}^1_s+a\mathbf{x}^1_{s,t},h_s))da)\mathbf{x}^1_{s,t}\otimes\mathbf{x}^1_{t,u}\\ &\quad+(d^{1,0}f(\mathbf{x}^1_s,h_s)-d^{1,0}f(\mathbf{x}^1_t,h_s))\mathbf{x}^2_{t,u}\\ &\quad+(f(\mathbf{x}^1_t,h_s)-f(\mathbf{x}^1_t,h_t))\mathbf{x}^1_{t,u}\\ &\quad+(d^{1,0}f(\mathbf{x}^1_t,h_s)-d^{1,0}f(\mathbf{x}^1_t,h_t))\mathbf{x}^2_{t,u}.\end{aligned}$$

Therefore, $(\mathbf{z}^1_{s,u}-\mathbf{z}^1_{s,t}-\mathbf{z}^1_{t,u})-(\widehat{\mathbf{z}}^1_{s,u}-\widehat{\mathbf{z}}^1_{s,t}-\widehat{\mathbf{z}}^1_{t,u})$ can be decomposed as the sum of the following eight terms

$$\begin{aligned}\Delta_1 &= \int_0^1((d^{1,0}f(\mathbf{x}^1_s,h_s)-d^{1,0}f(\mathbf{x}^1_s+a\mathbf{x}^1_{s,t},h_s))da)\mathbf{x}^1_{s,t}\otimes\mathbf{x}^1_{t,u}\\ &\quad-\int_0^1((d^{1,0}f(\widehat{\mathbf{x}}^1_s,h_s)-d^{1,0}f(\widehat{\mathbf{x}}^1_s+a\widehat{\mathbf{x}}^1_{s,t},h_s))da)\widehat{\mathbf{x}}^1_{s,t}\otimes\widehat{\mathbf{x}}^1_{t,u},\end{aligned}$$

$$\begin{aligned}\Delta_2 &= \left\{\int_0^1((d^{1,0}f(\widehat{\mathbf{x}}^1_s,h_s)-d^{1,0}f(\widehat{\mathbf{x}}^1_s+a\widehat{\mathbf{x}}^1_{s,t},h_s))da)\widehat{\mathbf{x}}^1_{s,t}\otimes\widehat{\mathbf{x}}^1_{t,u}\right\}\\ &\quad-\left\{\int_0^1((d^{1,0}f(\widehat{\mathbf{x}}^1_s,\widehat{h}_s)-d^{1,0}f(\widehat{\mathbf{x}}^1_s+a\widehat{\mathbf{x}}^1_{s,t},\widehat{h}_s))da)\widehat{\mathbf{x}}^1_{s,t}\otimes\widehat{\mathbf{x}}^1_{t,u}\right\},\end{aligned}$$

$$\begin{aligned}\Delta_3 &= \left\{(d^{1,0}f(\mathbf{x}^1_s,h_s)-d^{1,0}f(\mathbf{x}^1_t,h_s))\mathbf{x}^2_{t,u}\right\}\\ &\quad-\left\{(d^{1,0}f(\widehat{\mathbf{x}}^1_s,h_s)-d^{1,0}f(\widehat{\mathbf{x}}^1_t,h_s))\widehat{\mathbf{x}}^2_{t,u}\right\},\end{aligned}$$

$$\begin{aligned}\Delta_4 &= \left\{(d^{1,0}f(\widehat{\mathbf{x}}^1_s,h_s)-d^{1,0}f(\widehat{\mathbf{x}}^1_t,h_s))\widehat{\mathbf{x}}^2_{t,u}\right\}\\ &\quad-\left\{(d^{1,0}f(\widehat{\mathbf{x}}^1_s,\widehat{h}_s)-d^{1,0}f(\widehat{\mathbf{x}}^1_t,\widehat{h}_s))\widehat{\mathbf{x}}^2_{t,u}\right\},\end{aligned}$$

$$\Delta_5=\left\{(f(\mathbf{x}^1_t,h_s)-f(\mathbf{x}^1_t,h_t))\mathbf{x}^1_{t,u}\right\}-\left\{(f(\widehat{\mathbf{x}}^1_t,h_s)-f(\widehat{\mathbf{x}}^1_t,h_t))\widehat{\mathbf{x}}^1_{t,u}\right\},$$

$$\Delta_6=\left\{(f(\widehat{\mathbf{x}}^1_t,h_s)-f(\widehat{\mathbf{x}}^1_t,h_t))\widehat{\mathbf{x}}^1_{t,u}\right\}-\left\{(f(\widehat{\mathbf{x}}^1_t,\widehat{h}_s)-f(\widehat{\mathbf{x}}^1_t,\widehat{h}_t))\widehat{\mathbf{x}}^1_{t,u}\right\},$$

$$\Delta_7 = \left\{(d^{1,0}f(\mathbf{x}^1_t, h_s) - d^{1,0}f(\mathbf{x}^1_t, h_t))\mathbf{x}^2_{t,u}\right\} - \left\{(d^{1,0}f(\widehat{\mathbf{x}}^1_t, h_s) - d^{1,0}f(\widehat{\mathbf{x}}^1_t, h_t))\widehat{\mathbf{x}}^2_{t,u}\right\},$$
$$\Delta_8 = \left\{(d^{1,0}f(\widehat{\mathbf{x}}^1_t, \widehat{h}_s) - d^{1,0}f(\widehat{\mathbf{x}}^1_t, \widehat{h}_t))\widehat{\mathbf{x}}^2_{t,u}\right\} - \left\{(d^{1,0}f(\widehat{\mathbf{x}}^1_t, h_s) - d^{1,0}f(\widehat{\mathbf{x}}^1_t, h_t))\widehat{\mathbf{x}}^2_{t,u}\right\}.$$

Let us show how to bound $|\Delta_1|, |\Delta_4|$ and $|\Delta_5|$. The other bound will follow with similar methods. We decompose once again Δ_1 into $\Delta_{1,1} + \Delta_{1,2}$, where

$$\begin{aligned}\Delta_{1,1} =& \int_0^1 \left\{d^{1,0}f(\mathbf{x}^1_s, h_s) - d^{1,0}f(\mathbf{x}^1_s + a\mathbf{x}^1_{s,t}, h_s)\right\} \mathbf{x}^1_{s,t} \otimes \mathbf{x}^1_{t,u} da \\ &- \int_0^1 \left\{d^{1,0}f(\widehat{\mathbf{x}}^1_s, h_s) - d^{1,0}f(\widehat{\mathbf{x}}^1_s + a\widehat{\mathbf{x}}^1_{s,t}, h_s)\right\} \mathbf{x}^1_{s,t} \otimes \mathbf{x}^1_{t,u} da \\ =& \Delta_{1,1,1} - \Delta_{1,1,2} - \Delta_{1,1,3}\end{aligned}$$

with

$$\begin{aligned}\Delta_{1,1,1} =& \int_0^1 \int_0^1 d^{2,0}f(\widehat{\mathbf{x}}^1_s + b(\mathbf{x}^1_s - \widehat{\mathbf{x}}^1_s), h_s) \\ &\times ((\mathbf{x}^1_s - \widehat{\mathbf{x}}^1_s) \otimes \mathbf{x}^1_{s,t} \otimes \mathbf{x}^1_{t,u})\, da\, db, \\ \Delta_{1,1,2} =& \int_0^1 \int_0^1 d^{2,0}f(\widehat{\mathbf{x}}^1_s + a\widehat{\mathbf{x}}^1_{s,t} + b(\mathbf{x}^1_s - \widehat{\mathbf{x}}^1_s + a(\mathbf{x}^1_{s,t} - \widehat{\mathbf{x}}^1_{s,t})), h_s), \\ &\times ((\mathbf{x}^1_s - \widehat{\mathbf{x}}^1_s) \otimes \mathbf{x}^1_{s,t} \otimes \mathbf{x}^1_{t,u})\, da\, db \\ \Delta_{1,1,3} =& \int_0^1 \int_0^1 d^{2,0}f(\widehat{\mathbf{x}}^1_s + a\widehat{\mathbf{x}}^1_{s,t} + b(\mathbf{x}^1_s - \widehat{\mathbf{x}}^1_s + a(\mathbf{x}^1_{s,t} - \widehat{\mathbf{x}}^1_{s,t})), h_s), \\ &\times (a(\mathbf{x}^1_{s,t} - \widehat{\mathbf{x}}^1_{s,t}) \otimes \mathbf{x}^1_{s,t} \otimes \mathbf{x}^1_{t,u})\, da\, db\end{aligned}$$

and

$$\Delta_{1,2} = \int_0^1 \left\{d^{1,0}f(\widehat{\mathbf{x}}^1_s, h_s) - d^{1,0}f(\widehat{\mathbf{x}}^1_s + a\widehat{\mathbf{x}}^1_{s,t}, h_s)\right\} (\mathbf{x}^1_{s,t} \otimes \mathbf{x}^1_{t,u} - \widehat{\mathbf{x}}^1_{s,t} \otimes \widehat{\mathbf{x}}^1_{t,u}) da.$$

Let us first bound $|\Delta_{1,2}|$. As

$$\begin{aligned}\left|\mathbf{x}^1_{s,t} \otimes \mathbf{x}^1_{t,u} - \widehat{\mathbf{x}}^1_{s,t} \otimes \widehat{\mathbf{x}}^1_{t,u}\right| &= \left|(\mathbf{x}^2_{s,u} - \widehat{\mathbf{x}}^2_{s,u}) - (\mathbf{x}^2_{s,t} - \widehat{\mathbf{x}}^2_{s,t})\right| \\ &\leq 2\varepsilon R^2 \omega(s,u)^{2/p},\end{aligned}$$

$$\left|d^{1,0}f(\widehat{\mathbf{x}}^1_s, h_s) - d^{1,0}f(\widehat{\mathbf{x}}^1_s + a\widehat{\mathbf{x}}^1_{s,t}, h_s)\right| \leq \|F\|_{\mathrm{Lip}}\, aR\varepsilon\omega(s,u)^{1/p},$$

we obtain that $|\Delta_{1,2}| \leq K^1\omega(s,u)^\theta$, for $\theta > 1$ small enough. To bound $\Delta_{1,1}$, observe that $\Delta_{1,1,3}$ is bounded by $\|F\|_{\mathrm{Lip}} R^3\omega(s,u)^{3/p}$. Then, it is easy to see that $\left|(\mathbf{x}^1_s - \widehat{\mathbf{x}}^1_s) \otimes \mathbf{x}^1_{s,t} \otimes \mathbf{x}^1_{t,u}\right| \leq R^3\varepsilon\omega(0,u)^{1/p}\omega(s,u)^{2/p}$, and

$$\begin{aligned}\|F\|_{\mathrm{Lip}} R^{\gamma-2}\omega(s,u)^{(\gamma-2)/p} \geq& \Big|d^{2,0}f(\widehat{\mathbf{x}}^1_s + b(\mathbf{x}^1_s - \widehat{\mathbf{x}}^1_s), h_s) \\ &- d^{2,0}f(\widehat{\mathbf{x}}^1_s + a\widehat{\mathbf{x}}^1_{s,t} + b(\mathbf{x}^1_s - \widehat{\mathbf{x}}^1_s + a(\mathbf{x}^1_{s,t} - \widehat{\mathbf{x}}^1_{s,t})), h_s)\Big|.\end{aligned}$$

This implies that $|\Delta_{1,1,1} - \Delta_{1,1,2}|$ is bounded by

$$\|F\|_{\mathrm{Lip}} R^{\gamma+1}\omega(s,u)^{\gamma/p}\varepsilon\omega(0,u)^{1/p} = K^1\omega(s,u)^\theta.$$

Let us move on to Δ_4, which is the integral from $a=0$ to 1 of

$$\left\{d^{0,1}f(\widehat{\mathbf{x}}^1_s, h_s + a(\widehat{h}_s - h_s)) - d^{0,1}f(\widehat{\mathbf{x}}^1_t, h_s + a(\widehat{h}_s - h_s))\right\}(\widehat{h}_s - h_s)\otimes\widehat{\mathbf{x}}^2_{t,u}.$$

Therefore, it is bounded by $\|F\|_{\mathrm{Lip}} R^{\gamma}\omega(s,u)^{\gamma/p}\varepsilon\omega(0,u)^{1/p} \le K^2\varepsilon\omega(s,u)^\theta$.

Finally, Δ_5 is the sum of $\Delta_{5,1}+\Delta_{5,2}$, where $\Delta_{5,1}$ is the integral from $a=0$ to 1 of

$$\left\{d^{1,0}f(\mathbf{x}^1_t + a(\widehat{\mathbf{x}}^1_t - \mathbf{x}^1_t), h_s) - d^{1,0}f(\mathbf{x}^1_t + a(\widehat{\mathbf{x}}^1_t - \mathbf{x}^1_t), h_t)\right\}(\widehat{\mathbf{x}}^1_t - \mathbf{x}^1_t)\otimes\mathbf{x}^1_{t,u}.$$

Hence, $\Delta_{5,1}$ is bounded by $\|F\|_{\mathrm{Lip}} R^3\omega(s,u)^{1/p+1/q}\varepsilon\omega(0,u)^{1/p} \le K^3\omega(s,u)^\theta$. The other term $\Delta_{5,2}$ is

$$\Delta_{5,2} = (f(\widehat{\mathbf{x}}^1_t, h_s) - f(\widehat{\mathbf{x}}^1_t, h_t))(\widehat{\mathbf{x}}^1_{t,u} - \mathbf{x}^1_{t,u})$$

is bounded by $\|F\|_{\mathrm{Lip}} R^2\omega(s,u)^{1/p+1/q}\varepsilon \le K^3\varepsilon\omega(s,u)^\theta$.

Similarly to bound all the other Δ_i's, we find that

$$\left|(\mathbf{z}^1_{s,u} - \mathbf{z}^1_{s,t} - \mathbf{z}^1_{t,u}) - (\widehat{\mathbf{z}}^1_{s,u} - \widehat{\mathbf{z}}^1_{s,t} - \widehat{\mathbf{z}}^1_{t,u})\right| \le K^4\varepsilon\omega(s,u)^\theta.$$

To simplify the computation in bounding $\left|(\mathbf{z}^2_{s,u} - \mathbf{z}^2_{s,t} - \mathbf{z}^2_{t,u}) - (\widehat{\mathbf{z}}^2_{s,u} - \widehat{\mathbf{z}}^2_{s,t} - \widehat{\mathbf{z}}^2_{t,u})\right|$, we will assume that $f(\mathbf{x},h)$ does not depend on h. The general case would be done using techniques similar as the one just used, i.e. by separating the problems due to $\mathbf{x}$ and the one due to h. So, with this simplification, $-\mathbf{z}^2_{s,u} + \mathbf{z}^2_{s,t} + \mathbf{z}^2_{t,u}$ is equal to

$$\begin{aligned}
&\int_0^1 d^1 f^{\otimes 2}(\mathbf{x}^1_s + a\mathbf{x}^1_{s,t})\mathbf{x}^1_{s,t}\otimes\mathbf{x}^2_{t,u}da && \text{(line 1)}\\
&+\int_0^1 (f(\mathbf{x}^1_s)\otimes d^1 f(\mathbf{x}^1_s + a\mathbf{x}^1_{s,t}))(\mathbf{x}^1_{s,t}\otimes\mathbf{x}^1_{s,t}\otimes\mathbf{x}^1_{t,u})da && \text{(line 2)}\\
&+ f(\mathbf{x}^1_s)\otimes d^1 f(\mathbf{x}^1_t)\mathbf{x}^1_{s,t}\otimes\mathbf{x}^2_{t,u} && \text{(line 3)}\\
&+ d^1 f(\mathbf{x}^1_s)\otimes f(\mathbf{x}^1_t)\mathbf{x}^2_{s,t}\otimes\mathbf{x}^1_{t,u} && \text{(line 4)}\\
&+ d^1 f(\mathbf{x}^1_s)\otimes d^1 f(\mathbf{x}^1_t)\mathbf{x}^2_{s,t}\otimes\mathbf{x}^2_{t,u}. && \text{(line 5)}
\end{aligned}$$

Hence, $|(\mathbf{z}^2_{s,u} - (\mathbf{z}_{s,t}\otimes\mathbf{z}_{t,u})^2) - (\widehat{\mathbf{z}}^2_{s,u} - (\widehat{\mathbf{z}}_{s,t}\otimes\widehat{\mathbf{z}}_{t,u})^2)|$ can be bounded by $\widetilde{\Delta}_1 + \cdots + \widetilde{\Delta}_5$, where Δ_i is the difference of the i^{th} line in the above expression with the same expression replacing y and $\mathbf{x}$ by $\widehat{y}$ and $\widehat{\mathbf{x}}$.

$$\begin{aligned}
\widetilde{\Delta}_1 &= \left|\int_0^1 \left\{d^1 f(\mathbf{x}_s^1 + a\mathbf{x}_{s,t}^1) - d^1 f(\widehat{\mathbf{x}}_s^1 + a\widehat{\mathbf{x}}_{s,t}^1)\right\} \mathbf{x}_{s,t}^1 \otimes \mathbf{x}_{t,u}^2 da\right| \\
&\quad + \left|\int_0^1 d^1 f(\mathbf{x}_s^1 + a\mathbf{x}_{s,t}^1)(\mathbf{x}_{s,t}^1 \otimes \mathbf{x}_{t,u}^2 - \widehat{\mathbf{x}}_{s,t}^1 \otimes \widehat{\mathbf{x}}_{t,u}^2) da\right| \\
&\leq \|F\|_{\mathrm{Lip}}\, \varepsilon\omega(0,u)^{1/p}\omega(s,u)^{3/p} + \|F\|_{\mathrm{Lip}}\, \varepsilon\omega(s,u)^{3/p} \\
&\leq K^5 \varepsilon \omega(s,u)^\theta.
\end{aligned}$$

$$\begin{aligned}
\widetilde{\Delta}_2 &= \left|\int_0^1 (f(\mathbf{x}_s^1) \otimes d^1 f(\mathbf{x}_s^1 + a\mathbf{x}_{s,t}^1))(\mathbf{x}_{s,t}^1 \otimes \mathbf{x}_{s,t}^1 \otimes \mathbf{x}_{t,u}^1) da\right. \\
&\quad \left. - \int_0^1 (f(\widehat{\mathbf{x}}_s^1) \otimes d^1 f(\widehat{\mathbf{x}}_s^1 + a\widehat{\mathbf{x}}_{s,t}^1))(\mathbf{x}_{s,t}^1 \otimes \mathbf{x}_{s,t}^1 \otimes \mathbf{x}_{t,u}^1) da\right| \\
&\quad + \left|\int_0^1 (f(\widehat{\mathbf{x}}_s^1) \otimes d^1 f(\widehat{\mathbf{x}}_s^1 + a\widehat{\mathbf{x}}_{s,t}^1))((\mathbf{x}_{s,t}^1)^{\otimes 2} \otimes \mathbf{x}_{t,u}^1 - (\widehat{\mathbf{x}}_{s,t}^1)^{\otimes 2} \otimes \widehat{\mathbf{x}}_{t,u}^1) da\right| \\
&\leq \|F\|_{\mathrm{Lip}}\, \varepsilon\omega(0,u)^{1/p}\omega(s,u)^{3/p} + \|F\|_{\mathrm{Lip}}\, \varepsilon\omega(s,u)^{3/p} \\
&\leq K^5 \varepsilon \omega(s,u)^\theta.
\end{aligned}$$

Bounding $\widetilde{\Delta}_3 + \widetilde{\Delta}_4 + \widetilde{\Delta}_5$ is even easier, and we get that

$$|(\mathbf{z}_{s,u}^2 - (\mathbf{z}_{s,t} \otimes \mathbf{z}_{t,u})^2) - (\widehat{\mathbf{z}}_{s,u}^2 - (\widehat{\mathbf{z}}_{s,t} \otimes \widehat{\mathbf{z}}_{t,u})^2)| \leq K^6 \omega(s,u)^\theta.$$

That finishes the proof of the lemma. ∎

We can now prove Proposition 5.

Proof of Proposition 5. For a fixed $0 \leq s \leq t \leq T$, it is easy to see that

$$|z_{s,t}^1 - \widehat{z}_{s,t}^1| \leq K^7 \varepsilon \omega(s,t)^{1/p} \text{ and } |z_{s,t}^2 - \widehat{z}_{s,t}^2| \leq K^8 \varepsilon \omega(s,t)^{2/p}.$$

From this inequality, the previous lemma and Theorem 6 in the appendix, we therefore obtain that

$$d_{p,\omega,T}\left(\int f(x,h) d\mathbf{x}, \int f(\widehat{x},\widehat{h}) d\widehat{\mathbf{x}}\right) \leq K\varepsilon,$$

which concludes the proof. ∎

Proposition 6. *Let g be a $\mathrm{Lip}((\alpha-1,\beta-1), V_1 \times V_2 \to L(V_2,W))$ one-form with*

$$\alpha > \frac{pq+q-p}{q} \text{ and } \beta > q.$$

Then the map $(\mathbf{x},h) \mapsto \int g(x,h) dh$ is locally Lipschitz continuous from $C^{(p,q),\omega}([0,T], T^2(V_1) \times V_2)$ into $C^{q,\omega}([0,T],W)$.

The proof is very similar, and somewhat easier, to the one of Proposition 5, so we leave it to the reader.

We now switch to the construction of solutions of differential equations.

4 Solving a Differential Equation

We denote by V_1, V_2, W_1, W_2 some separable Banach spaces. We define $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$.

Let $(\mathbf{x}, h)$ be a (p, q)-rough path in $C^{(p,q),\omega}([0, T], T^2(V_1) \times V_2)$, with $p \in [2, 3)$, $q \geq 1$ and $p^{-1} + q^{-1} > 1$. As in [Lyo98, LQ02], we now want to use our integral to define solutions of differential equations driven by the (p, q)-rough path $(\mathbf{x}, h)$.

4.1 Notion of a Solution

Let A be a linear map from V_1 into functions from W into W_1, i.e. $(dv_1, w) \to A(w)(dv_1)$ is linear in dv_1, and for a fixed dv_1, $A(.)dv_1$ is a function from W into W_1. We also consider a linear map B from V_2 into functions from W into W_2.

Example 2. A typical example is given by d functions $A_1, \cdots, A_d$ from $W_1 \oplus W_2$ into W_1, where $d = \dim V_1$. Then if $e_1, \ldots, e_d$ is a basis of V_1, we can set $A(y, k)(\sum_{i=1}^d e_i dx^i) = \sum_{i=1}^d A_i(y, k)dx^i$.

We want to solve integral equations of the type:

$$\left\{ \begin{array}{l} \mathbf{y}_{0,t} = \int_0^t A(y_s, k_s)d\mathbf{x}_s, \\ k_{0,t} = \int_0^t B(y_s, k_s)dh_s. \end{array} \right. \tag{8}$$

The class of differential equations that we consider is actually more general than it looks.

Example 3. The following integral equation

$$\left\{ \begin{array}{l} \mathbf{y}_{0,t} = \int_0^t A(y_s, k_s)d\mathbf{x}_s + \int_0^t C(y_s, k_s)dh_s, \\ k_{0,t} = \int_0^t B(y_s, k_s)dh_s. \end{array} \right. \tag{9}$$

are actually a special case of the equation of the type (8). Indeed, consider the following system of equations:

$$\left\{ \begin{array}{l} \widetilde{y}_{0,t} = \int_0^t \widetilde{A}(\widetilde{y}_s, \ell_s, k_s)d\mathbf{x}_s, \\ \ell_{0,t} = \int_0^t \widetilde{C}(\widetilde{y}_s, \ell_s, k_s)dh_s, \\ k_{0,t} = \int_0^t \widetilde{B}(\widetilde{y}_s, \ell_s, k_s)dh_s, \end{array} \right.$$

where $\widetilde{A}(y, \ell, k) = A(y + \ell, k)$, $\widetilde{B}(y, \ell, k) = B(y + \ell, k)$, and $\widetilde{C}(y, \ell, k) = C(y + \ell, k)$. Then $(\widetilde{y}_t + \ell_t, k_t)$ solves the differential equation (9).

When one thinks to the case of semi-martingales, it is natural to think of equations such as the first one in (9). The second line of (9) is given at no real supplementary cost. This term is also added, if $B = C$, to keep the spirit of a "stability" theorem: a (p,q)-rough path is transformed into a (p,q)-rough path in the same way that a semi-martingale is transformed into a semi-martingale when one solves SDEs.

What is important in our approach is that we need less regularity for C and B than for A. As we will see below, this allows us to consider differential equations driven by rough path that are not geometric without strengthening the regularity assumptions on A, B and C.

From the maps A and B, we define a map F_A from $V_1 \oplus W_1 \oplus W_2$ to $L(V_1 \oplus W_1, V_1 \oplus W_1)$ and F_B from $V_1 \oplus W_1 \oplus W_2$ to $L(V_2, V_2 \oplus W_2)$ by

$$\begin{aligned} F_A(v_1, w_1, w_2)(dv_1, dw_1) &= (dv_1, A(w_1, w_2)dv_1), \\ F_B(v_1, w_1, w_2)(dv_2) &= (dv_2, B(w_1, w_2)dv_2). \end{aligned}$$

We then define a map F from $X = V_1 \oplus W_1 \oplus V_2 \oplus W_2$ to $L(X, X)$, by

$$\begin{aligned} &F(v_1, w_1, v_2, w_2)(dv_1, dw_1, dv_2, dw_2) \\ &\qquad = (F_A(v_1, w_1, w_2)(dv_1, dw_1), F_B(v_1, w_1, w_2)dv_2). \end{aligned}$$

Note that $\int_0^t F(z_u, h_u, k_u)d(\mathbf{z}_u, h_u, k_u) = \left(\int_0^t F_A(z_u, k_u)d\mathbf{z}_u, \int_0^t F_B(z_u, k_u)dh_u\right)$. We now define, in the same spirit as in the classical rough path theory, the meaning of solution of a differential equation with rough driving signals.

Definition 4. We fix some $T_1 \in (0, T]$. By a solution of (8) on $[0, T_1]$, we mean a rough path $(\mathbf{z}, h, k)$ in $C^{(p,q),\omega}([0, T_1], T^2(V_1 \oplus W_1) \times (V_2 \oplus W_2))$ such that for $t \in [0, T_1]$,

$$\begin{aligned} (\mathbf{z}_{0,t}, h_{0,t}, k_{0,t}) &= \left(\int_0^t F_A(z_u, k_u)d\mathbf{z}_u, \int_0^t F_B(z_u, k_u)dh_u\right), \qquad (10) \\ (\mathbf{z}_0, h_0, k_0)^1 &= ((x_0, y_0), h_0, k_0). \end{aligned}$$

and such that $\mathbf{z}$ projects onto $\mathbf{x} \in C^{p,\omega}([0, T_1], T^2(V_1))$.

For $T_1 \in (0, T]$, we define the map Ψ_{T_1} from $C^{(p,q),\omega}([0, T_1], T^2(V_1 \oplus W_1) \times (V_2 \oplus W_2))$ into itself by

$$\Psi_{T_1}(\mathbf{z}, h, k) = \int F(\mathbf{z}_u, h_u, k_u)d(\mathbf{z}_u, h_u, k_u).$$

A solution of the differential equation (11) therefore corresponds to a fixed point of our application Ψ_{T_1} which projects onto $(\mathbf{x}, h) \in T^2(V_1) \times V_2$. The space $C^{(p,q),\omega}([0, T_1], T^2(V_1 \oplus W_1) \times (V_2 \oplus W_2))$ is equipped with the topology defined by the distance $d_{\omega,(p,q),T_1}$, where

$$d_{\omega,(p,q),T_1}((\mathbf{z}, h, k), (\widetilde{\mathbf{z}}, \widetilde{h}, \widetilde{k})) = \max\left\{d_{\omega,p,T_1}(\mathbf{z}, \widetilde{\mathbf{z}}), d_{\omega,q,T_1}(h, \widetilde{h}), d_{\omega,q,T_1}(k, \widetilde{k})\right\}.$$

We also define $\|(\mathbf{z}, h, k)\|_{\omega,(p,q),T_1} = \max\left\{\|\mathbf{z}\|_{\omega,p,T_1}, \|h\|_{\omega,q,T_1}, \|k\|_{\omega,q,T_1}\right\}$.

4.2 Existence

We first give a compactness result.

In this section, $(\mathbf{x},h)$ is a fixed element of (p,q)-rough path in the space $C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2)$, and of course, that fixes $\|\mathbf{x}\|_{p,\omega,T}$, $\|h\|_{p,\omega,T}$. For $R>0$ and $T_1\in(0,T]$, we denote by $B^{(\mathbf{x},h)}_{(p,q),\omega,T_1}(0,R)$ the set of elements $(\mathbf{z},h,k)$ in the space $C^{(p,q),\omega}([0,T_1],T^2(V_1\oplus W_1)\times(V_2\oplus W_2))$ that projects onto $(\mathbf{x},h)$ and such that $\|(\mathbf{z},h,k)\|_{(p,q),\omega,T_1}\leq R$.

Lemma 4. *For all $R>R_0=\max\left\{\|B\|^q_{\mathrm{Lip}}\|h\|^q_{p,\omega,T},\|A\|^p_{\mathrm{Lip}}\|\mathbf{x}\|^p_{p,\omega,T}\right\}$, there exists $0<T_1\leq T$, such that*

$$\Psi_{T_1}(B^{(\mathbf{x},h)}_{(p,q),\omega,T_1}(0,R))\subset B^{(\mathbf{x},h)}_{(p,q),\omega,T_1}(0,R).$$

Proof. Let (E,h,D) be the almost multiplicative functional with values in $T^2(V_1\oplus W_1)$ defined by

$$\begin{aligned}
E^1_{s,t}&=F_A(y_s,k_s)\mathbf{x}^1_{s,t}+d^{1,0}F_A(y_s,k_s)\mathbf{z}^2_{s,t},\\
E^2_{s,t}&=F_A(y_s,k_s)\otimes F_A(y_s,k_s)\mathbf{x}^2_{s,t},\\
h_{s,t}&=h_{s,t},\\
D_{s,t}&=F_B(y_s,k_s)h_{s,t},
\end{aligned}$$

so that $\Psi_T(\mathbf{z},h,k)$ is the (p,q)-rough path constructed from to E,h,D. More precisely, $\widehat{\mathbf{z}}$ (resp. $\widehat{k}$), the projection of $\Psi_T(\mathbf{z},h,k)$ onto $T^2(V_1\oplus W_1)$ (resp. onto W_2) is $\mathfrak{M}(E)$ (resp. $\mathfrak{M}(D)$). There exists a constant K depending (only) on p, q, θ, $\omega(0,T)$, $\|\mathbf{x}\|_{p,\omega,T}$, $\|h\|_{p,\omega,T}$, $\|A\|_{\mathrm{Lip}}$, $\|B\|_{\mathrm{Lip}}$ and R such that for all $0\leq s\leq t\leq T_1$,

$$\|\Psi_{T_1}(\mathbf{z},h,k)_{s,t}-(E_{s,t},D_{s,t})\|\leq K\omega(0,T_1)^\theta\text{ with }\theta>1.$$

Let us remark that

$$\|d^{1,0}F_A(y_s,k_s)\mathbf{z}^2_{s,t}\|\leq\|A\|_{\mathrm{Lip}}R^{2/p}\omega(s,t)^{2/p}.$$

Thus, for all $0\leq s\leq t\leq T_1$,

$$\|E^1_{s,t}\|\leq\|A\|_{\mathrm{Lip}}\omega(s,t)^{1/p}(\|\mathbf{x}\|_{p,\omega,T}+R^{2/p}\omega(0,T_1)^{1/p}).$$

On the other hand,

$$\|E^2_{s,t}\|\leq\|A\|^2_{\mathrm{Lip}}\|\mathbf{x}\|^2_{p,\omega,T}\omega(s,t)^{2/p}.$$

Thus,

$$\|\widehat{\mathbf{z}}^1_{s,t}\|\leq\omega(s,t)^{1/p}(K\omega(0,T_1)^{\theta-1/p}+\|A\|_{\mathrm{Lip}}(\|\mathbf{x}\|_{p,\omega,T}+R^{2/p}\omega(0,T_1)^{1/p}))\quad(11)$$

and

$$\|\widehat{\mathbf{z}}^2_{s,t}\| \leq \omega(s,t)^{2/p}(K\omega(0,T_1)^{\theta-2/p} + \|A\|^2_{\mathrm{Lip}}\|\mathbf{x}\|^2_{p,\omega,T}\omega(0,T_1)^{2/p}). \qquad (12)$$

In a similar way,

$$\|\widehat{k}_{s,t}\| \leq \omega(s,t)^{1/q}(K\omega(0,T_1)^{\theta-1/q} + \|B\|_{\mathrm{Lip}}\|h\|_{p,\omega,T}).$$

For any $R > R_0$, as $\lim_{T_1\to 0}\omega(0,T_1) = 0$, there exists a choice of T_1 small enough such that

$$\max\left\{\begin{array}{c} K\omega(0,T_1)^{\theta-1/q} + \|B\|_{\mathrm{Lip}}\|h\|_{p,\omega,T}, \\ K\omega(0,T_1)^{\theta-1/p} + \|A\|_{\mathrm{Lip}}(\|\mathbf{x}\|_{p,\omega,T} + R^{2/p}\omega(0,T_1)^{1/p}), \\ \sqrt{K\omega(0,T_1)^{\theta-2/p} + \|A\|^2_{\mathrm{Lip}}\|\mathbf{x}\|^2_{p,\omega,T}\omega(0,T_1)^{2/p}} \end{array}\right\} \leq R^{1/p}.$$

Thus, when R and T_1 are fixed,

$$\|\Psi_{T_1}(\mathbf{z},h,k)\|_{\omega,(p,q),T_1} \leq R.$$

which concludes the proof. ∎

Theorem 1. *Let us assume that the dimensions of W_1 and W_2 are finite. Let $(\mathbf{x},h)$ in $C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2)$. Under the assumptions of Lemma 4, there exists at least one solution to* (10) *in $C^{(p,q),\omega}([0,T],T^2(V_1\oplus W_1)\times(V_2\oplus W_2))$ on $[0,T]$.*

Ideally, we would apply just a fixed point theorem to Ψ_T. Unfortunately, there are no fixed point theorem that we know we can apply here directly. Nevertheless, by a trick, we are able to apply Schauder's fixed point theorem. The element $(\mathbf{x},h)$ in $C^{(p,q),\omega}([0,T],T^2(V_1)\times V_2)$ is fixed.

We first define B_T to be the space of continuous maps (y,y^x) from $\{0 \leq s < t \leq T\}$ into $W_1 \oplus (W_1\otimes V_1)$, such that for all $0 \leq s < t \leq T$,
(i) $y_{s,t} = y_{0,t} - y_{0,s}$,
(ii) $y^x_{s,t} = y^x_{0,t} - y^x_{0,s} + y^x_{0,s}\otimes x_{s,t}$,
(iii) $|y_{s,t}| + \left|y^x_{s,t}\right|^{1/2} \leq K\omega(s,t)^{1/p}$ for a constant K.

The space B_T is then clearly a vector space, and with the norm

$$\|(y,y^x)\|_{B_T} = \sup_{0\leq s<t\leq T}\max\left\{\frac{|y_{s,t}|}{\omega(s,t)^{1/p}}, \frac{\left|y^x_{s,t}\right|}{\omega(s,t)^{2/p}}\right\}.$$

it is easy to see that B_T becomes a Banach space.

Let π be the canonical projection from $G^2(V_1\oplus W_1)$ onto $V_1\oplus(W_1\otimes V_1)$. Then, if $\mathbf{z}$ and $\mathbf{z}'$ are elements in $C^{p,\omega}([0,T],T^2(V_1\oplus W_1))$ that both project onto $\mathbf{x}$ and such that $\pi(\mathbf{z}) = \pi(\mathbf{z}')$, then $\Psi_T(\mathbf{z}, h, k) = \Psi_T(\mathbf{z}', h, k)$. Indeed, the multiplicative functional used to construct Ψ only uses $\pi(\mathbf{z})$ and k (this being true as $(\mathbf{x}, h)$ is given). One can then define $\widetilde{\Psi}_T$ from $B_T \times C^{q,\omega}([0,T], W_2)$ into $C^{(p,q),\omega}([0,T_1], T^2(V_1 \oplus W_1) \times (V_2 \oplus W_2))$, such that for all $\mathbf{z} \in C^{p,\omega}([0,T], T^2(V_1 \oplus W_1))$, $\widetilde{\Psi}_T(\pi(\mathbf{z}), k) = \Psi(\mathbf{z}, h, k)$. Moreover, working as in Lemma 4, there exists R_0 such that for all $R > R_0$, there exists $0 < T_1 \leq T$, such that the projection of $\widetilde{\Psi}_{T_1}$ onto $B_{T_1} \times C^{q,\omega}([0,T_1], W_2)$ sends the ball of radius R (for $\|.\|_{B_{T_1}}$) into itself. If the spaces W_1 and W_2 are of finite dimension, then this ball is compact. By Schauder's fixed point theorem, there exists an element $((y, y^x), k)$ such that the projection of $\widetilde{\Psi}_{T_1}((y, y^x), k)$ onto $B \times C^{q,\omega}([0,T], W_2)$ is equal to $((y, y^x), k)$. In other words, $\widetilde{\Psi}_{T_1}((y, y^x), k)$ is a solution (up to time T_1) of our rough differential equation. The solution up to time T is then obtained by concatenation, as in [LQ02]. ∎

4.3 Uniqueness and Continuity

The signal $(\mathbf{x}, h)$ in $C^{(p,q),\omega}([0,T], T^2(V_1) \times V_2)$ is once again fixed. We fix stronger assumptions on the differential forms we integrate, and we can then prove the existence of a solution to (8) using a contraction fixed point theorem, which allows to drop the hypotheses of the finiteness of the dimensions of W_1 and W_2. A continuity result with respect to the path $(\mathbf{x}, h)$ is also deduced.

Lemma 5. *We assume that A is a linear map from V_1 into* $\mathrm{Lip}((\gamma, \kappa), W_1 \oplus W_2 \to W_1)$*-functions with*

$$\gamma > p \text{ and } \kappa > \frac{qp + p - q}{p},$$

and that B is a linear map from V_2 into $\mathrm{Lip}((\alpha, \beta), W_1 \oplus W_2 \to W_2)$*-functions with*

$$\alpha > \frac{pq + q - p}{q} \text{ and } \beta > q.$$

Then, for all $R > 0$, there exists $T_2 > 0$ such that Ψ_{T_2} restricted to $B^{(\mathbf{x},h)}_{(p,q),\omega,T_2}(0, R)$ is a contraction of parameter $\frac{1}{2}$.

Proof. We fix a $R > 0$, some time $T_2 \in [0, T]$ (to be determined later) and two elements $(\mathbf{z}, h, k)$ and $(\widehat{\mathbf{z}}, h, \widehat{k})$ in $B^{(\mathbf{x},h)}_{(p,q),\omega,T_2}(0, R)$, and we define

$$\varepsilon = d_{(p,q),\omega,T_2}((\widehat{\mathbf{z}}, h, \widehat{k}), (\mathbf{z}, h, k)).$$

We denote by $K^i(T_2)$ some functions of time T_2 (which might depend on $\|A\|_{\mathrm{Lip}}$, $\|B\|_{\mathrm{Lip}}$, $\omega(0,T)$, R, p, q, γ, κ, β), which decrease to 0 when T_2 goes

to 0.
We recall that

$$\Psi_{T_2}(\mathbf{z},h,k)_{s,t} = \left(\int_s^t F_A(z_u,k_u)d\mathbf{z}_u, \int_s^t F_B(z_u,k_u)dh_u\right),$$

We will write y (resp. $\widehat{y}$) for the path which is the projection of $\mathbf{z}$ (resp. $\widehat{\mathbf{z}}$) onto W_1.

$$\begin{aligned} E^1_{s,t} &= F_A(y_s,k_s)\mathbf{z}^1_{s,t} + d^{1,0}F_A(y_s,k_s)\mathbf{z}^2_{s,t}, \\ E^2_{s,t} &= (F_A(y_s,k_s)\otimes F_A(y_s,k_s))\mathbf{z}^2_{s,t}, \end{aligned}$$

is the almost multiplicative functional used to construct $\int F_A(\mathbf{z}_u,k_u)d\mathbf{z}_u$, and we define similarly $\widehat{E}_{s,t}$ to be the almost multiplicative functional used to construct $\int F_A(\widehat{\mathbf{z}}_u,\widehat{k}_u)d\widehat{\mathbf{z}}_u$. By Lemma 3, for all $0\leq s\leq t\leq u\leq T_2$,

$$\begin{aligned} |(E^1_{s,u}-E^1_{s,t}-E^1_{t,u}) - (\widehat{E}^1_{s,u}-\widehat{E}^1_{s,t}-\widehat{E}^1_{t,u})| &\leq K\varepsilon\omega(s,u)^\theta, \\ |(E^2_{s,u}-(E_{s,t}\otimes E_{t,u})^2) - (\widehat{E}^2_{s,u}-(\widehat{E}_{s,t}\otimes \widehat{E}_{t,u})^2)| &\leq K\varepsilon\omega(s,u)^\theta. \end{aligned}$$

Taking $1<\widetilde{\theta}=\frac{1+\theta}{2}<\theta$, we see that we can write

$$|(E^1_{s,u}-E^1_{s,t}-E^1_{t,u}) - (\widehat{E}^1_{s,u}-\widehat{E}^1_{s,t}-\widehat{E}^1_{t,u})| \leq K^1(T_2)\varepsilon\omega(s,u)^{\widetilde{\theta}} \quad (13)$$

$$|(E^2_{s,u}-(E_{s,t}\otimes E_{t,u})^2) - (\widehat{E}^2_{s,u}-(\widehat{E}_{s,t}\otimes \widehat{E}_{t,u})^2)| \leq K^1(T_2)\varepsilon\omega(s,u)^{\widetilde{\theta}} \quad (14)$$

Moreover, $E^1_{s,t}=F_A(y_s,k_s)\mathbf{x}^1_{s,t}+d^{1,0}F_A(y_s,k_s)\mathbf{z}^2_{s,t}$. Hence

$$\begin{aligned} |E^1_{s,t}-\widehat{E}^1_{s,t}| \leq\ & |F_A(y_s,k_s)-F_A(\widehat{y}_s,\widehat{k}_s)|\cdot|\mathbf{x}^1_{s,t}| \\ &+|d^{1,0}F_A(y_s,k_s)\mathbf{z}^2_{s,t}-d^{1,0}F_A(\widehat{y}_s,\widehat{k}_s)\mathbf{z}^2_{s,t}| \\ &+|d^{1,0}F_A(\widehat{y}_s,\widehat{k}_s)(\mathbf{z}^2_{s,t}-\widehat{\mathbf{z}}^2_{s,t})|. \end{aligned} \quad (15)$$

We therefore see that

$$\begin{aligned} |E^1_{s,t}-\widehat{E}^1_{s,t}| &\leq \|F\|_{\mathrm{Lip}} R^2\varepsilon \max\left\{\omega(0,T_2)^{1/p},\omega(0,T_2)^{1/q}\right\}\omega(s,t)^{1/p} \\ &\quad + \|F\|_{\mathrm{Lip}} R^2\varepsilon \max\left\{\omega(0,T_2)^{1/p},\omega(0,T_2)^{1/q}\right\}\omega(s,t)^{2/p} \\ &\quad + \|F\|_{\mathrm{Lip}} R^2\varepsilon\omega(s,t)^{2/p} \\ &\leq K^2(T_2)\varepsilon\omega(s,t)^{1/p}. \end{aligned}$$

Similarly,

$$|E^2_{s,t}-\widehat{E}^2_{s,t}|\leq K^3(T_2)\varepsilon\omega(s,t)^{1/p}. \quad (16)$$

From inequalities (13)–(16), and Theorem 6 in the appendix, we obtain that

$$d_{p,\omega,T_2}\left(\int F_A(\widehat{z}_u,\widehat{k}_u)d\widehat{\mathbf{z}}_u,\int F_A(z_u,k_u)d\mathbf{z}_u\right)\leq K^4(T_2)d_{(p,q),\omega,T_2}((\widehat{\mathbf{z}},h,\widehat{k}),(\mathbf{z},h,k)).$$

An analogous argument shows that

$$d_{p,\omega,T_2}\left(\int F_B(\widehat{z}_u,\widehat{k}_u)d\widehat{h}_u,\int F_B(z_u,k_u)dh_u\right)\leq K^5(T_2)d_{(p,q),\omega,T_2}((\widehat{\mathbf{z}},h,\widehat{k}),(\mathbf{z},h,k)).$$

This implies that

$$d_{(p,q),\omega,T_2}(\Psi_{T_2}(\widehat{\mathbf{z}},h,\widehat{k}),\Psi_{T_2}(\mathbf{z},h,k))\leq K^6(T_2)d_{(p,q),\omega,T_2}((\widehat{\mathbf{z}},h,\widehat{k}),(\mathbf{z},h,k)).$$

In particular, taking $T_2>0$ small enough, we obtain that

$$d_{(p,q),\omega,T_2}(\Psi_{T_2}(\widehat{\mathbf{z}},h,\widehat{k}),\Psi_{T_2}(\mathbf{z},h,k))\leq \frac{1}{2}d_{(p,q),\omega,T_2}((\widehat{\mathbf{z}},h,\widehat{k}),(\mathbf{z},h,k)).$$

This proves the Lemma. ∎

Theorem 2. *Let* $(\mathbf{x},h)$ *in* $C^{(p,q),\omega}([0,T_1],T^2(V_1)\times V_2)$. *We assume that* A *is a linear map from* V_1 *into* $\mathrm{Lip}((\gamma,\kappa),W_1\oplus W_2\to W_1)$*-functions with*

$$\gamma>p,\ \text{and}\ \kappa>\frac{qp+p-q}{p},$$

and that B *is a linear map from* V_2 *into* $\mathrm{Lip}((\alpha,\beta),W_1\oplus W_2\to W_2)$*-functions with*

$$\alpha>\frac{pq+q-p}{q}\ \text{and}\ \beta>q.$$

Then, there exists a unique solution to (10) *in* $C^{(p,q),\omega}([0,T],T^2(V_1\oplus W_1)\times(V_2\oplus W_2))$. *We denote by* $I_{A,B}(\mathbf{x},h)$ *this solution.*

Proof. We have already seen the existence[1]. If $(\mathbf{z},h,k)$ and $(\widehat{\mathbf{z}},h,\widehat{k})$ are two solutions, then we have that for T_2 small enough,

$$d_{(p,q),\omega,T_2}((\widehat{\mathbf{z}},h,\widehat{k}),(\mathbf{z},h,k))\leq\frac{1}{2}d_{(p,q),\omega,T}((\widehat{\mathbf{z}},h,\widehat{k}),(\mathbf{z},h,k)),$$

i.e. $(\widehat{\mathbf{z}},h,\widehat{k})=(\mathbf{z},h,k)$ up to time T_2. We obtain uniqueness over the whole interval by, once again, concatenation. ∎

[1] Note that the extra smoothness assumption is a trivial corollary of the previous lemma and the contraction fixed point theorem.

Theorem 3. *Under the assumptions of Lemma 5, the map*

$$\begin{array}{rcl} C^{(p,q),\omega}([0,T_1],T^2(V_1)\times V_2) & \to & C^{(p,q),\omega}([0,T],T^2(V_1\oplus W_1)\times(V_2\oplus W_2)) \\ (\mathbf{x},h) & \to & I_{A,B}(\mathbf{x},h) \end{array}$$

is continuous.

Proof. Let $(\mathbf{z}_0,h,k_0)_{s,t}=(\mathbf{x}_{s,t},h_{s,t},0)$ be in $C^{(p,q),\omega}([0,T],T^2(V_1\oplus W_1)\times(V_2\oplus W_2))$. We define a Picard iteration by:

$$\begin{aligned} (\mathbf{z}_{n+1},h,k_{n+1}) &= \Psi_T(\mathbf{z}_n,h,k_n) \\ &= \Psi_T^{\circ(n+1)}(\mathbf{z}_0,h,k_0). \end{aligned}$$

By Lemma 4, there exists R such that for all $n\geq 0$,

$$\|(\mathbf{z}_n,h,k_n)\|_{(p,q),\omega,T}\leq R.$$

Then, by Lemma 5, there exists $0<T_2\leq T$ such that

$$d_{(p,q),\omega,T_2}(I_{A,B}(\mathbf{x},h),(\mathbf{z}_n,h,k_n))\leq 2^{-n}d_{(p,q),\omega,T_2}((\mathbf{z}_1,h,k_1),(\mathbf{z}_0,h,k_0)).$$

Moreover, for all n, the map $\Psi_{T_2}^{\circ n}$is continuous, being the composition of n continuous (and even locally Lipschitz) functions. Hence, by a 3ε-type argument, we obtain the continuity of the map $I_{A,B}$. ∎

4.4 A Corollary

A special case of the above is the following corollary.

Corollary 1. *We fix $p\geq 1$ and $\varepsilon>0$. Let A_1 be a linear map from V into* Lip$(p+\varepsilon,W\to W)$ *functions, and A_0 a* Lip$(1+\varepsilon,W\to W)$ *function from W into itself. Given a V-valued smooth path x, consider the differential equation*

$$dy_t=A_0(y_t)dt+A_1(y_t)dx_t.$$

Then the map $I_{(A_0,A_1)}:x\to(x,y)$ extends uniquely to a continuous map from $(C^{p,\omega}([0,T],T^2(V)),\|\cdot\|_{p,\omega,T})$ into $(C^{p,\omega}([0,T],T^2(V\oplus W)),\|\cdot\|_{p,\omega,T})$, whenever the $\omega(s,t)\geq K(t-s)$, i.e. if the 1-variation of the map $t\to t$ is controlled by ω.

5 General Itô Formula

5.1 p-Rough Paths as a Pair of Geometric Rough Paths

We start with a simple result on $T^2(V)$, which will lead us see that non-geometric rough paths can be interpreted as a pair of geometric ones. Obviously, $(\mathrm{Sym}(V^{\otimes 2}),+)$ is a group (a commutative one) and is a subgroup of $T^2(V)$. In this Section, we assume only that $p \geq 2$, but this result concerns only paths with values in $T^2(V)$, which is then not sufficient to consider integrating differential forms if $p \geq 3$.

Proposition 7. *Let*

$$\begin{array}{rcl}\Upsilon^{-1} : G^2(V) \times \mathrm{Sym}(V^{\otimes 2}) & \longrightarrow & T^2(V) \\ (g,w) & \longrightarrow & g + w = g \otimes w,\end{array}$$

$$\begin{array}{rcl}\Upsilon_1 : T^2(V) & \longrightarrow & G^2(V) \\ (v_1,v_2) & \longrightarrow & \exp(v_1 + \mathrm{Anti}(v_2)),\end{array}$$

and

$$\begin{array}{rcl}\Upsilon_2 : T^2(V) & \longrightarrow & \mathrm{Sym}(V^{\otimes 2}) \\ (v_1,v_2) & \longrightarrow & \mathrm{Sym}(v_2) - \dfrac{1}{2}v_1^{\otimes 2}.\end{array}$$

Then Υ^{-1} is a group isomorphism from the product group $G^2(V)\times\mathrm{Sym}(V^{\otimes 2})$ onto $T^2(V)$, with inverse given by (Υ_1,Υ_2). Moreover, $\|\cdot\|$ and $\|\Upsilon_1(\cdot)\| + |\Upsilon_2(\cdot)|^{1/2}$ are equivalent homogeneous norm on $T^2(V)$.

Proof. First note that $\mathrm{Sym}(V^{\otimes 2})$ is actually a subgroup of the center of $T^2(V)$. Therefore,

$$\begin{array}{rcl}\Upsilon^{-1}((g_1\otimes g_2, w_1+w_2)) & = & g_1\otimes g_2\otimes w_1\otimes w_2 \\ & = & g_1\otimes w_1\otimes g_2\otimes w_2 \\ & = & \Upsilon^{-1}(g_1,w_1)\otimes\Upsilon^{-1}(g_2,w_2),\end{array}$$

which proves that Υ^{-1} is a group homomorphism. It is easy to check that $\Upsilon^{-1}\circ(\Upsilon_1,\Upsilon_2)$ (resp. $(\Upsilon_1,\Upsilon_2)\circ\Upsilon^{-1})$) is the identity map of $T^2(V)$ (resp. $G^2(V)\times\mathrm{Sym}(V^{\otimes 2})$). Finally, observing that for all $g\in T^2(V)$, $g = \Upsilon_1(g)\otimes\Upsilon_2(g)$, we see that

$$\begin{array}{rcl}\|g\| & \leq & \|\Upsilon_1(g)\| + \|\Upsilon_2(g)\| \\ & = & \|\Upsilon_1(g)\| + \sqrt{|\Upsilon_2(g)|}.\end{array}$$

For $v_2 \in V^{\otimes 2}$, our symmetry assumption on $|\cdot|$ tells us that there exists c such that

$$|v_2| \leq |\mathrm{Sym}(v_2)| + |\mathrm{Anti}(v_2)| \leq c\,|v_2|\,.$$

In particular,

$$\begin{aligned}\frac{1}{2}\left\|\Upsilon_2(v_1,v_2)\right\|^2 &= \left|\mathrm{Sym}(v_2) - \frac{1}{2}v_1^{\otimes 2}\right| \\ &\leq |\mathrm{Sym}(v_2)| + \frac{1}{2}\left|v_1\right|^2 \leq c\,|v_2| + \frac{1}{2}\left|v_1\right|^2 \\ &\leq c'\left\|(v_1,v_2)\right\|^2.\end{aligned}$$

Similarly,

$$\left\|\Upsilon_1(v_1,v_2)\right\| \leq c'\left\|(v_1,v_2)\right\|,$$

which ends up the proof of equivalence of homogeneous norm. ∎

This immediately gives a one-to-one correspondence between $C^{p\text{-var}}([0,T],T^2(V))$ and the product space $C^{p\text{-var}}([0,T],G^2(V)) \times C^{p/2\text{-var}}([0,T],\mathrm{Sym}(V^{\otimes 2}))$.

Corollary 2. *Let $\mathbf{x} \in C^{p,\omega}([0,T],T^2(V))$, i.e. $\mathbf{x}$ is a p-rough path controlled by ω. Then,*

$$\begin{aligned}\Upsilon_1(\mathbf{x}) : [0,1] &\longrightarrow G^2(V) \\ t &\longrightarrow \Upsilon_1(\mathbf{x}_t),\end{aligned}$$

belongs to $C^{p,\omega}([0,T],G^2(V))$, i.e. is a weak geometric p-rough path, and

$$\begin{aligned}\Upsilon_2(\mathbf{x}) : [0,1] &\longrightarrow \mathrm{Sym}(V^{\otimes 2}) \\ t &\longrightarrow \Upsilon_2(\mathbf{x}_t)\end{aligned}$$

belongs to $C^{p/2,\omega}([0,T],\mathrm{Sym}(V^{\otimes 2}))$, i.e. is a weak geometric $p/2$-rough path. Reciprocally, if $(\mathbf{y},\psi) \in C^{p,\omega}([0,T],G^2(V)) \times C^{p/2,\omega}([0,T],\mathrm{Sym}(V^{\otimes 2}))$ then $\mathbf{x}_t = \mathbf{y}_t + \psi_t = \Upsilon^{-1}(\mathbf{y}_t,\psi_t) \in C^{p,\omega}([0,T],T^2(V))$, i.e. it is a p-rough path controlled by ω.

The rough path $\Upsilon_1(\mathbf{x})$ is the geometric rough path constructed from the area of $\mathbf{x}$ (that is the antisymmetric part of $\mathbf{x}^2$). We therefore see that we can identify $C^{p\text{-var}}([0,T],T^2(V))$ with $C^{p\text{-var}}([0,T],G^2(V)) \times C^{p/2\text{-var}}([0,T],\mathrm{Sym}(V^{\otimes 2}))$ thanks to the map

$$\mathbf{x} \rightarrow (\Upsilon_1(\mathbf{x}),\Upsilon_2(\mathbf{x})).$$

In other words, we have establish a bijection between p-rough paths and weak geometric $(p,p/2)$-rough paths.

Example 4. We are given a V-valued semi-martingale M, with quadratic variation process $\langle M, M\rangle$ (seen as a path of bounded variation with values in $V\otimes V$). Let $\Pi^n = \{\,0 \leq t_1^n \leq \cdots \leq t_k^n \leq T\,\}$ be a deterministic subdivision of $[0,T]$, with $\sup(t_{i+1}^n - t_i^n) \xrightarrow[n\to\infty]{} 0$. For any $\lambda\in[0,1]$ and any $t\in[0,T]$, we set

$$\int_0^t Y_s \otimes d_\lambda M_s = \lim_{n\to\infty} \sum_{i \text{ s.t. } t_{i+1}^n \leq t} (\lambda Y_{t_i^n} + (1-\lambda) Y_{t_{i+1}^n})(M^j_{t_{i+1}^n} - M^j_{t_i^n}),$$

whenever the the limit (in probability) exists. When the parameter $\lambda = 1$, we obtain Itô's integral, for $\lambda = \frac{1}{2}$ the Stratonovich's integral and finally for $\lambda = 0$, the backward Itô's integral.

It is classical that whenever f is a C^1 one-form,

$$\int_0^t f(M_s) d_\lambda M_s = \int_0^t f(M_s) d_{1/2} M_s + \left(\frac{1}{2} - \lambda\right) \int_0^t df(M_s) \mathrm{d}_{1/2} \left\langle M, M\right\rangle_s \quad (17)$$

We define the λ-lift of M to a p-rough path:

$$\mathbf{M}_t^\lambda = \left(M_t, \int_0^t (M_r - M_0)\otimes d_\lambda M_r\right).$$

We will use the notation $\mathbf{M}_t^{1/2} = \mathbf{M}_t^{\mathrm{Strat}}$ and $\mathbf{M}_t^1 = \mathbf{M}_t^{\mathrm{Itô}}$. The p-rough path $\mathbf{M}_t^{\mathrm{Strat}}$ is a geometric one, but for all $\lambda \in [0,1]\setminus\left\{\frac{1}{2}\right\}$, $\mathbf{M}_t^\lambda$ is not a geometric p-rough path, but just a p-rough path ($2<p<3$). Moreover, by equation (17), it is easy to see that, a.s., for all $\lambda\in[0,1]$,

$$\begin{aligned} \Upsilon_1(\mathbf{M}^\lambda) &= \mathbf{M}^{\mathrm{Strat}}, \\ \Upsilon_2(\mathbf{M}^\lambda) &= \left(\frac{1}{2}-\lambda\right)\langle M, M\rangle . \end{aligned}$$

5.2 Itô's Formula

The following generalised Itô formula described how to related the integration of one-form along a p-rough path $\mathbf{x}$ and along its associated geometric p-rough path. In particular, it relates the integration along $\mathbf{M}^{\mathrm{Itô}}$ and $\mathbf{M}^{\mathrm{Strat}}$, i.e. is a generalisation of Itô's formula. Such a generalisation at the level of rough paths already appeared in [LQ96, LQ02]. The following form of Itô's formula at the level of rough path seems to the authors clearer.

Theorem 4. *Let* $\mathbf{x}$ *in* $C^{p,\omega}([0,T],T^2(V))$*, and* f *a* $\mathrm{Lip}(\gamma-1, V\to L(V,W))$ *one-form, with* $\gamma>p$ *and* $p\in[2,3)$*. We define the one-forms*

$$\begin{aligned} f_1 : V\oplus \mathrm{Sym}(V^{\otimes 2}) &\longrightarrow L(V\oplus\mathrm{Sym}(V^{\otimes 2}), W) \\ (v_1, v_2) &\longrightarrow ((dv_1, dv_2)\to (f(v_1)dv_1 + df(v_1)dv_2)) \end{aligned}$$

and

$$\begin{aligned} f_2 : V \oplus \mathrm{Sym}(V^{\otimes 2}) &\longrightarrow L(V \oplus \mathrm{Sym}(V^{\otimes 2}), \mathrm{Sym}(W^{\otimes 2})) \\ (v_1, v_2) &\longrightarrow ((dv_1, dv_2) \to (f(v_1) \otimes f(v_1)) dv_2). \end{aligned}$$

Then, $\int f(x)d\mathbf{x}$ belongs to $C^{p,\omega}([0,T], T^2(W))$ and it decomposes into an element of $C^{p,\omega}([0,T], G^2(W)) \times C^{p/2,\omega}([0,T], \mathrm{Sym}(V^{\otimes 2}))$ in the following way:

$$\Upsilon_1\left(\int f(x)d\mathbf{x}\right) = \int f_1(x)d(\Upsilon_1(\mathbf{x}), \Upsilon_2(\mathbf{x})), \tag{18}$$

$$\Upsilon_2\left(\int f(x)d\mathbf{x}\right) = \int f_2(x)d\Upsilon_2(\mathbf{x}). \tag{19}$$

Such integrals make sense by the result of the first section.

Proof. The rough path $\int f(x)d\mathbf{x}$ is the p-rough path associated to the almost multiplicative functional

$$\begin{aligned} z^1_{s,t} &= f(x_s)\mathbf{x}^1_{s,t} + df(x_s)\mathbf{x}^2_{s,t}, \\ z^2_{s,t} &= f(x_s) \otimes f(x_s)\mathbf{x}^2_{s,t}. \end{aligned}$$

In particular, we obtain that $\Upsilon_1(\int f(x)d\mathbf{x})$ is the p-geometric rough path associated to almost multiplicative functional $\Upsilon_1(z_{s,t})$, with

$$\begin{aligned} \Upsilon_1(z_{s,t})^1 &= f(x_s)\mathbf{x}^1_{s,t} + df(x_s)\mathbf{x}^2_{s,t} = f(x_s)\mathbf{x}^1_{s,t} + df(x_s)\Upsilon_1(\mathbf{x}_{s,t})^2 + df(x_s)\Upsilon_2(\mathbf{x}_{s,t}) \\ \Upsilon_1(z_{s,t})^2 &= f(x_s) \otimes f(x_s)\Upsilon_1(\mathbf{x}_{s,t})^2. \end{aligned}$$

As this is precisely the almost multiplicative that we use to construct $\int f_1(x)d(\Upsilon_1(\mathbf{x}), \Upsilon_2(\mathbf{x}))$, we obtain Equation (18). The rough path $\Upsilon_2(\int f(x)d\mathbf{x})$ is the $p/2$-rough path associated to

$$\begin{aligned} \Upsilon_2(z_{s,t}) &= f(x_s) \otimes f(x_s)\,\mathrm{Sym}(\mathbf{x}^2_{s,t}) - \frac{1}{2}(f(x_s) \otimes f(x_s))(\mathbf{x}^1_{s,t})^{\otimes 2} \\ &\quad - \frac{1}{2}(f(x_s) \otimes df(x_s))\mathbf{x}^1_{s,t} \otimes \mathbf{x}^2_{s,t} - \frac{1}{2}(df(x_s) \otimes f(x_s))\mathbf{x}^2_{s,t} \otimes \mathbf{x}^1_{s,t} \\ &\quad - \frac{1}{2}(df(x_s) \otimes df(x_s))\mathbf{x}^2_{s,t} \otimes \mathbf{x}^2_{s,t} \\ &= f(x_s) \otimes f(x_s)\Upsilon_2(\mathbf{x}_{s,t}) - \frac{1}{2}(df(x_s) \otimes df(x_s))\mathbf{x}^2_{s,t} \otimes \mathbf{x}^2_{s,t} \\ &\quad - \frac{1}{2}(f(x_s) \otimes df(x_s))\mathbf{x}^1_{s,t} \otimes \mathbf{x}^2_{s,t} - \frac{1}{2}(df(x_s) \otimes f(x_s))\mathbf{x}^2_{s,t} \otimes \mathbf{x}^1_{s,t}. \end{aligned}$$

As

$$\begin{aligned} \Big| df(x_s) \otimes df(x_s)\,\mathbf{x}^2_{s,t} \otimes \mathbf{x}^2_{s,t} + f(x_s) \otimes df(x_s)\,\mathbf{x}^1_{s,t} \otimes \mathbf{x}^2_{s,t} & \\ + df(x_s) \otimes f(x_s)\,\mathbf{x}^2_{s,t} \otimes \mathbf{x}^1_{s,t} \Big| &\leq K\omega(s,t)^{3/p}, \end{aligned} \tag{20}$$

with $3/p > 1$, we see that the rough path associated to the almost multiplicative functional $\Upsilon_2(z_{s,t})$ and the one associated to the almost multiplicative functional $f(x_s) \otimes f(x_s)\Upsilon_2(\mathbf{x}_{s,t})$ are the same. The first one is $\Upsilon_2(\int f(x)d\mathbf{x})$, while the second one is $\int f_2(x)d\Upsilon_2(\mathbf{x})$. ∎

We come back to our Example 4, using the same notations.

Example 5. From the very early development of the rough paths theory [Sip93], it is known that the rough path integral $\int_0^t f(M_r)d\mathbf{M}_r^{1/2}$ corresponds at the first level to the Stratonovich integral $\int_0^t f(M_r) \circ dM_r$ for Hölder continuous martingales. This is usually seen as a consequence of the Wong-Zakai theorem. This result is also true for all continuous martingales [CL05]. By Formula (18) projected onto the first level in the first line, the identification of Stratonovich integration and rough path integration with respect to $\mathbf{M}^{\mathrm{Strat}}$ in the second line, and equation (17) in the third line, we obtain that, when f is a $C^{1+\varepsilon}$ one-form

$$\begin{aligned}
\left(\int f(M_s)d\mathbf{M}_s^{\lambda}\right)^1 &= \left(\int f(M_s)d\mathbf{M}_s^{\mathrm{Strat}}\right)^1 + \left(\frac{1}{2}-\lambda\right)\int df(M_s)d\left\langle M,M\right\rangle_s \\
&= \int f(M_s)\circ dM_s + \left(\frac{1}{2}-\lambda\right)\int df(M_s)d\left\langle M,M\right\rangle_s \\
&= \int f(M_s)d_{\lambda}M_s.
\end{aligned}$$

We have therefore proved that the stochastic d_λ integration corresponds to the rough path integration along $\mathbf{M}^{\lambda}$.

Example 6. Let $g : V \to W$ a $C^{2+\varepsilon}$ function. Then, looking only at the W-component of Equation (18) with $f = dg$, we obtain

$$g(M_t) - g(M_0) = \left(\int_0^t dg(M_s)d\mathbf{M}_s^{It\hat{o}}\right)^1 + \frac{1}{2}\int_0^t d^2g(M_s)d\left\langle M,M\right\rangle_s$$

which is precisely Itô's formula. Equation (19) reads in this semi-martingale context

$$\left\langle \left(\int_0^{\cdot} f(M_s)d\mathbf{M}_s^{\mathrm{It\hat{o}}}\right)^1, \left(\int_0^{\cdot} f(M_s)d\mathbf{M}_s^{\mathrm{It\hat{o}}}\right)^1 \right\rangle_t = \int_0^t f(M_s)^{\otimes 2}d\left\langle M,M\right\rangle_s,$$

which is the well known formula to compute the quadratic variation of an Itô integral.

5.3 Approximation of Non-Geometric Rough Paths by Smooth Paths

The following is a straightforward consequence of the identification of the space of p-rough path with the space of weak geometric $(p, p/2)$-rough paths, and of Theorem 16 in [FV04].

Theorem 5. *The rough path* $\mathbf{y}$ *belongs to* $C^{p\text{-var}}([0,T], T^2(V))$ *with* $p \geq 1$ *if and only if there exists a sequence of infinitely differentiable* V*-valued paths* $y_1(n)$ *and* $\mathrm{Sym}(V^{\otimes 2})$*-valued path* $y_1(n)$ *such that*
(i) *the* p*-variation of the canonical lift* $\mathbf{y}_1(n)$ *of* $y_1(n)$ *to a* $G^2(V)$*-valued path and the* $p/2$*-variation of* $y_2(n))$ *are uniformly bounded in* n*.*
(ii) $\mathbf{y}_1(n)$ *converges pointwise to* $\Upsilon_1(\mathbf{y})$ *and* $y_2(n)$ *converges pointwise to* $\Upsilon_2(\mathbf{y})$*.*

Note that (i) and (ii) is actually equivalent to the fact that the p-variation of $\Upsilon^{-1}(\mathbf{y}_1(n), y_2(n))$ is uniformly bounded, and that $\Upsilon^{-1}(\mathbf{y}_1(n), y_2(n))$ converges pointwise to $\mathbf{y}$. Moreover, this implies that for all $q > p$, $\Upsilon^{-1}(\mathbf{y}_1(n), y_2(n))$ converges in the q-variation metric to $\mathbf{y}$.

Moreover, by the continuity of the integration of a one-form, if f is a $\mathrm{Lip}(\gamma - 1, V \to L(V, W))$ one-form with $\gamma > p$, then

$$\int f(y_1(n)_s) d\Upsilon^{-1}(\mathbf{y}_1(n), y_2(n))$$

converges in the q-variation metric to $\int f(\mathbf{y}_s^1) \mathrm{d}\mathbf{y}_s$. Note that the above integral is a Riemann integral, but involves f and its differential df. A similar continuity statements obviously holds for differential equations.

Remark 1. Define $C^{0,p\text{-var}}([0,T], T^2(V))$ to be the set of elements Y in $C^{p\text{-var}}([0,T], T^2(V))$ such that there exists a sequence of infinitely differentiable V-valued paths $y_1(n)$ and $\mathrm{Sym}(V^{\otimes 2})$-valued path $y_1(n)$ satisfying
(i) the canonical lift of $y_1(n)$ to a $G^2(V)$-valued path converges in the p-variation metric to $\Upsilon_1(Y)$.
(ii) $y_2(n)$ converges in the $p/2$-variation metric to $\Upsilon_2(Y)$.

Then, $C^{0,p\text{-var}}([0,T], T^2(V))$ is a closed subset of $C^{p\text{-var}}([0,T], T^2(V))$, and it is a Polish space [FV04]. Elements of $C^{0,p\text{-var}}([0,T], T^2(V))$ which take their values in $G^2(V)$ form precisely the set of geometric p-rough paths (in V).

A Appendix

Theorem 6. *Let* X, Y *be two almost rough paths of roughness* p *in* $T^{\lfloor p \rfloor}(V)$*, both controlled by a given* ω*, i.e. for all* $0 \leq s < t < u \leq T$*,* $i = 1, \cdots, \lfloor p \rfloor$*,*

$$\max\{|(X_{s,t} \otimes X_{t,u})^i - X_{s,u}^i|, |(Y_{s,t} \otimes Y_{t,u})^i - Y_{s,u}^i|\} \leq M\omega(s,u)^\theta,$$

for some constant $M_1, M_2 \geq 0$ *and* $\theta > 1$. *We assume moreover that for all* $0 \leq s < t < u \leq T$, $i = 1, \cdots, \lfloor p \rfloor$,

$$|X^i_{s,t} - Y^i_{s,t}| \leq \varepsilon\omega(s,t)^{i/p},$$
$$|((X_{s,t} \otimes X_{t,u})^i - X^i_{s,u}) - ((Y_{s,t} \otimes Y_{t,u})^i - Y^i_{s,u})| \leq \varepsilon\omega(s,u)^{\theta}.$$

Let $\widehat{X}$ *and* $\widehat{Y}$ *the rough paths associated to* X *and* Y. *Then, there exists a constant* K *depending only on* $\theta, p, \|X\|_{p,\omega}, \|Y\|_{q,\omega}, M, \omega(0,T)$ *such that*

$$|\widehat{X}^i_{s,t} - \widehat{Y}^i_{s,t}| \leq K\varepsilon\omega(s,t)^{i/p},$$

for all $i = 1, \cdots, \lfloor p \rfloor$ *and* $0 \leq s < t \leq T$.

Proof. The proof is just as in theorem 3.2.1 in [LQ02]. ∎

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